Impartial-culture asymptotics

a central limit theorem for manipulation of elections

Geoffrey Pritchard*, Mark Wilson

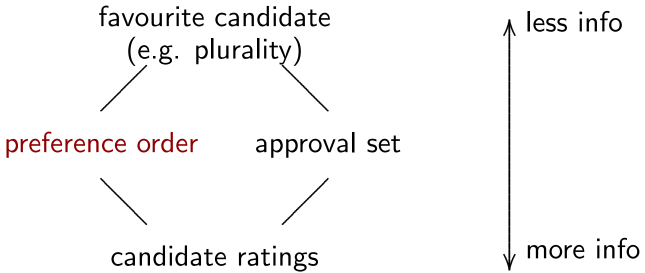
University of Auckland

March 20, 2009

Voting rules

- One of m candidates must be elected by n voters.

How much information to ask the voters for?



Preference-order rules

- Each voter has one of the $m!$ possible preference orders (types, opinions).
- A full *profile* specifies the type of each voter.
- A *voting situation* specifies only the number of voters of each type
  - this is all we need, if the voting rule treats voters symmetrically (anonymously).
- e.g. 3 candidates, 6 preference orders

$$N = (N_1, N_2, N_3, N_4, N_5, N_6), \qquad \text{with } \sum_{i=1}^{6} N_i = n$$

Scoring (positional) voting rules

A candidate gets w_i points when a voter ranks him in ith place;

$$1 = w_1 \geq w_2 \geq \cdots \geq w_m = 0.$$

Example (3 candidates):	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
number of voters N_t:	2	2	0	3	1	0

- For $w = (1, \frac{1}{2}, 0)$ (Borda's rule), a wins.
- For $w = (1, 1, 0)$ (anti-plurality rule), c wins.

Probabilistic voter behaviour

- IAC: all voting situations are equally likely to occur.
    - For large n, our voting situation is approximately uniformly distributed on a simplex.
    - Probabilities $\longrightarrow$ volumes of convex bodies...
- IC: voters have independent, uniform random types.
    - For large n, our voting situation is approximately (multivariate) normally distributed.
    - Central Limit Theorem, here we come...

IC asymptotics

- Voting situation

$$N_t \;\dot{\sim}\; \frac{n}{m!} + \sqrt{n}\frac{(m!-1)^{1/2}}{m!} Z_t, \qquad Z_t \sim N(0,1)$$

  - The voter types are about equally numerous.

- Scoreboard

$$|\alpha| = \sum_t N_t \sigma_t(\alpha) \;\dot{\sim}\; n\bar{w} + \sqrt{n}\sigma_w \left(\frac{m}{m-1}\right)^{1/2} (Z_\alpha - \bar{Z})$$

  - The scores tend to be nearly equal.

Tied scores

Ignore the possibility of tied scores.

$$P\,(\text{any ties}) \to 0 \qquad \text{as } n \to \infty.$$

Manipulation

- Logical possibility of manipulation: some coalition of voters can improve the result (for themselves) by voting insincerely.
  - Ignores counterthreats
  - Ignores complexity
- IC is very manipulable:

$$P\,(\text{L.P.M.}) \longrightarrow 1 \qquad \text{as } n \longrightarrow \infty$$

  for all scoring rules except anti-plurality.
- Minimum manipulating coalition size MCS (∞ if not L.P.M.)
  - Study the distribution of this random variable.

Recruiting a manipulating coalition

- Our coalition will contain (for each type t):
  x_t voters (sincerely) of type t;
  y_t voters who insincerely vote t;

$$\sum_t x_t = \sum_t y_t.$$

- Post-manipulation score of α is

$$|\alpha| + \sum_t (y_t - x_t)\sigma_t(\alpha).$$

Manipulation: an integer linear program

Minimum manipulating coalition size $MCS = \min_\beta Q_1(\beta)$, where

$$
\begin{aligned}
Q_1(\beta) = \min_{x,y} \sum_t x_t & & \\
\text{s.t.} \quad |\beta| + \sum_t (y_t - x_t)\sigma_t(\beta) &\geq |\alpha| + \sum_t (y_t - x_t)\sigma_t(\alpha) \quad \forall \alpha \neq \beta \\
\sum_t x_t &= \sum_t y_t \\
y_t &\geq 0 \\
0 &\leq x_t \leq N_t \\
x_t, y_t & \quad \text{integer}
\end{aligned}
$$

For IC and large n, we'll want $x_t \sim \sqrt{n}$, but $N_t \sim n$,
so the last two constraints will very rarely matter.

Phantom voters

Let Q_2 = min. coalition size *without* the last two constraints.
Now we can recruit non-existent voters, of any types we please.

Example (3 candidates):	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
number of voters N_t:	2	2	0	3	1	0

Borda scores: $|a| = 4.5$, $|b| = 4$, $|c| = 3.5$.

- Regular manipulation: $Q_1(b) = \infty$.
  - Everybody who prefers b to a already ranks b top, a bottom.
- Relaxed manipulation: $Q_2(b) = 1$.
  - One *cba* could do it (by voting *bca*).
  - To make b sole winner, 1.00001 such voters would suffice.

But this example is misleading...

Phantom voters don't hurt

Theorem. Relaxing makes manipulation easier, but not by much.

$$P\left(|Q_1(\beta) - Q_2(\beta)| \leq K\right) \to 1 \qquad \text{as } n \to \infty,$$

where K depends only on the voting rule.

- Coalition sizes $Q_i(\beta) \sim \sqrt{n}$, so allowing phantom voters really hasn't made much difference.

Phantom-voter manipulation is well-behaved

- **Theorem.** Second-placegetter has smallest phantom manipulating coalition.

$$\min_\beta Q_2(\beta) = Q_2(b).$$

(Only the constraint $x_t \geq N_t$ could have given another candidate a smaller one.)

- **Theorem.** Minimal phantom coalition for b consists only of types

$$\ldots ba \ldots$$

(They can insincerely put b first and a last.)

Phantom-voter manipulation is well-behaved

- **Theorem.** Second-placegetter has smallest phantom manipulating coalition.

$$\min_\beta Q_2(\beta) = Q_2(b).$$

(Only the constraint $x_t \geq N_t$ could have given another candidate a smaller one.)

- **Theorem.** Minimal phantom coalition for b consists only of types

$$\ldots ba \ldots$$

(They can insincerely put b first and a last.)

An even simpler linear program

- Recruit z_i phantom voters of types ranking b in ith place, a in $(i+1)$st place.

  Consider

$$
\begin{aligned}
Q = \min_z \sum_i z_i & & & \\
\text{s.t.} \quad \sum_i (1 - w_i + w_{i+1}) z_i &\geq |a| - |b| & & (b \text{ catches up to } a) \\
\sum_i (1 - w_i) z_i &\geq n\bar{w} - |b| & & (b \text{ above average}) \\
z_i &\geq 0 & &
\end{aligned}
$$

- **Theorem.** These two constraints are enough!

$$Q = Q_2(b) \quad (\approx MCS).$$

A two-variable linear program

- Take the dual linear program: two variables only.

$$Q = \max\{(|a| - n\bar{w})\lambda + (n\bar{w} - |b|)\mu : (\lambda, \mu) \in M_w\}$$

where the feasible set

$$M_w = \{(\lambda, \mu) : 0 \le \lambda \le \mu \text{ and } w_{i+1}\lambda + (1 - w_i)\mu \le 1 \ \forall i\}$$

depends only on the voting rule.

- The random coefficients

$$(|a| - n\bar{w}, n\bar{w} - |b|) \approx \text{ bivariate normal}$$

Asymptotic behaviour of MCS

Theorem.

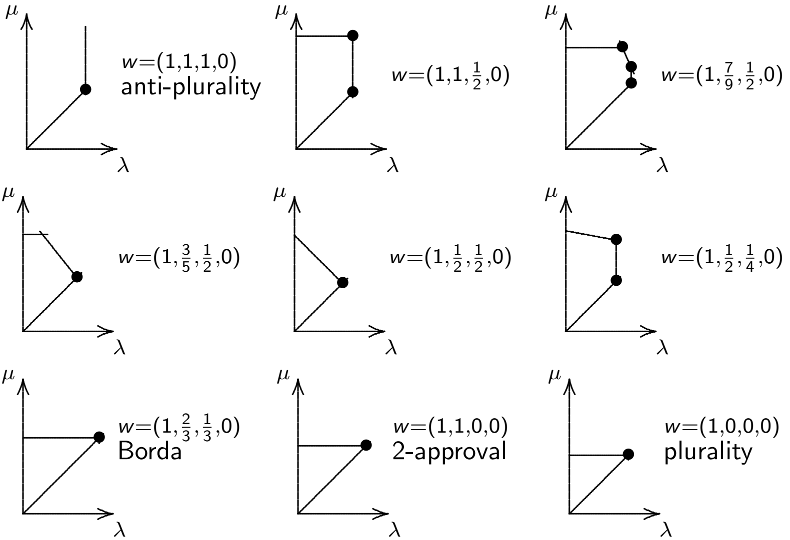
$$\frac{MCS}{\sqrt{n}} \xrightarrow{D} V_w, \qquad \text{i.e. } P\left(MCS \le v\sqrt{n}\right) \approx P\left(V_w \le v\right)$$

where

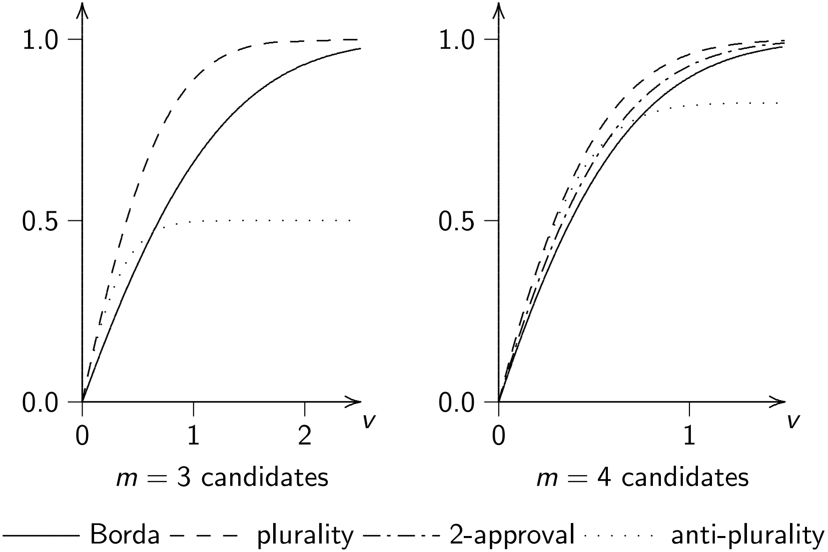
$$V_w = \max\left\{\lambda(\rho_1(Z) - \bar{Z}) + \mu(\bar{Z} - \rho_2(Z)) : (\lambda, \mu) \in \sigma_w \left(\frac{m}{m-1}\right)^{1/2} M_w\right\}$$

and $\rho_1(Z)$, $\rho_2(Z)$ are the two largest among m standard normal variables.

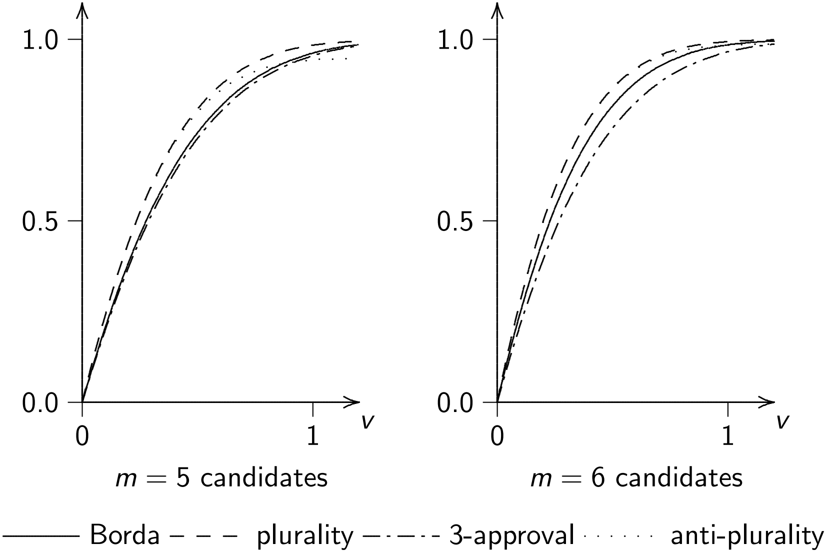
Four-candidate voting rules: the feasible sets $\sigma_w M_w$



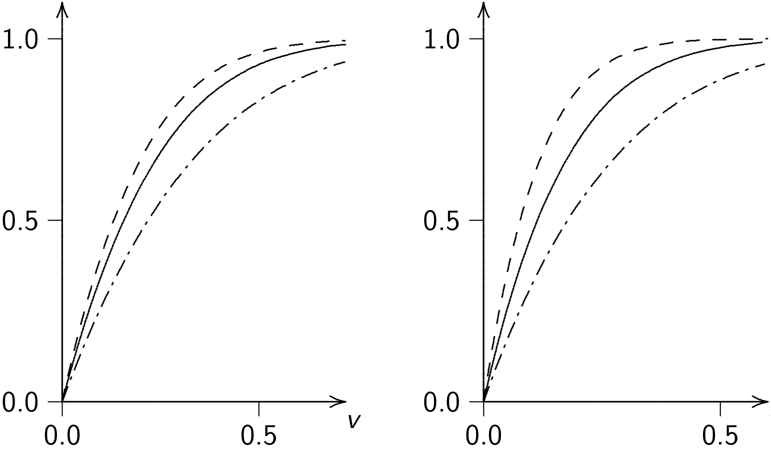
P (manipulability by some coalition of size $\leq v\sqrt{n}$)



$P\left(\text{manipulability by some coalition of size } \leq v\sqrt{n}\right)$



$P\left(\text{manipulability by some coalition of size } \leq v\sqrt{n}\right)$



$m = 10$ candidates

$m = 20$ candidates

——— Borda   – – – plurality   –·–·· $m/2$-approval