DESIGNING A STUDY PROCESS OF THE CENTRAL LIMIT THEOREM FOR ENGINEERS

Hugo Alvarado
Universidad Católica de la Santísima Concepción, Chile
Carmen Batanero
Universidad de Granada, Spain
alvaradomartinez@ucsc.cl

In this paper we analyse the meaning planned by a teacher to introduce the Central Limit Theorem to engineers. This meaning takes into account the nature of variables and different approaches to the distribution of the sum or random variables. The planning of teaching is based on the mathematical knowledge about the theorem and the theoretical framework about semiotic functions, which is currently been developed at the University of Granada, Spain. The different elements of meaning for this theorem were characterised after the analysis of a sample of University statistics textbooks, the study of the theorem evolution along history and previous research related to its understanding. The didactic proposal incorporates simulation with manipulative materials and the @Risk *Software (Risk analyser for Microsoft* Excel*). Results from this work will serve to improve the correct application of different elements of meaning for central limit theorem when solving engineering problems and to prepare new didactic proposals to teach statistics to engineers.*

INTRODUCTION

The Central Limit Theorem (CLT) is fundamental in the applications of statistics in engineering, both in providing approximations to many probability distributions for some values of its parameters and in building asymptotic sampling distributions for the mean and other parameters in big samples and for a variety of conditions. Understanding of this theorem, however, rests in the previous learning of many basic concepts, such as random variable and distribution, sampling distribution, and convergence. Under the expression "central limit theorem" we can unfold a variety of meanings, depending of which of the different theorems that deal with the limit of the distribution for the sum of random variables we refer, among those theorems, which have been deduced by different mathematicians, along more than 200 years. Today we either present students with only a finished version of CLT, where we mix several of the original theorems or we do not discuss with them the conditions for its correct application. Moreover, in many cases it is not possible to give a complete proof of some of these theorems (the original proofs being too complicate for the students) and lack of time forces us to skip too quickly to applications.

Some researchers (e.g., Méndez, 1991) carried out assessment on how students understand some of the basic features of simple versions of TCL (e.g., the sum of independent identically distributed random variables). In other cases (e.g., delMas, Garfield and Chance, 1999) they organized teaching experiments to see how students could develop an intuitive understanding of sampling distribution deduced from TCL. In our research we take a different approach, Starting from the history of statistics we first characterise the different historical TCL meanings, each of them of increasing complexity. We then compare these historical meanings with the different approaches to the TCL in statistics textbooks used in the teaching of statistics to engineers in Chile. Finally, we propose a didactical approach to introduce the TCL to engineers taking into account the previous analyses with the aim of improving the teaching of statistics in engineering. All of this is part of a wider research project where we plan to experiment and assess the didactical proposal.

We base on a theoretical model on the meaning of mathematical objects (Godino and Batanero, 1998; Godino, 2002) and we identify what the authors call elements of TCL meaning: a) fields of problems that gave origin to the TCL; verbal, symbolic and graphical representations use to work with these problems and to represent the different mathematical objects involved; c) procedures used by different authors to solve the problems in the problem field; mathematical objects and properties used in these procedures, and e) arguments used to justify the solutions.

BRIEF HISTORICAL SKETCH OF THE CENTRAL LIMIT THEORETICAL DEVELOPMENT

A look to the TCL history helps us find different stages in the TCL development linked to different problem fields (see Fisher, 2000; Xiuyu, 2003):

1. *Approximating the binomial distribution B(n,p) for increasing values of n*. Abraham De Moivre published in 1733 "Doctrine of Chances," where he included a proof for the normal approximation to the Binomial distribution,
2. *Finding the distribution of the limit of the sum of n random variables under some conditions*. Since binomial distribution can be seen as the sum of *n* identical independently distributed Bernoulli distributions, other authors later generalises the previous result: In 1809 Laplace gave a proof of CLT for mutually independent, identically distributed discrete random variables with finite mean and variance; in 1824 Poisson extended the result to continuous random variables. Cauchy in 1853 deduced the result for i.i.d continuous symmetric bounded random variables; Chebyscheff (1887) generalized the theorem for distributions with infinite range, provided some moments are finite; Later the theorem was expanded to non identically distributions or dependent variables.
3. Some other authors were worried about the *precision of the estimation* (e.g., Dirichlet or Cauchy).
4. A problem related to all above was finding sufficient and complete condition for the theorem validity; Lindeberg, Lévy, Feller and others provided these conditions.

All this history suggests the theorem arose little by little from a very particular problem (approximation of Binomial distribution) what was successively generalised. In developing the theorem, the above authors provided statistical and mathematical tools that were later used in other applications of statistics, such as characteristic and generator functions, moments method, etc. The TCL is today used in a variety of indirect applications, such as finding asymptotic sampling distribution for different parameters, building confidence intervals or testing hypotheses for these parameters or finding an adequate sample size to carry out an inference with a given precision.

MEANING OF CENTRAL LIMIT THEOREM IN STATISTICAL BOOKS FOR ENGINEERS

Once we finished the characterization of different historical fields of problems in the development of the central limit theorem we carried out the analysis of a sample of 16 statistics textbooks that are used in the teaching of the introductory statistical course for engineers in Chile. These textbooks were selected after consulting the books recommended in official programmes in Chile Universities. Following our theoretical model (Godino, 2002) we analysed each of the different elements of meaning of TCL in the textbooks and came to the following conclusions:

Problems Related to TCL

Most problems presented in the textbooks refer to applications of TCL (secondary problems fields) in sampling and inference. We believe that some few examples of the original problem fields from where the theorem arose would be useful for students grasp the full TCL meaning. Manipulative materials (such as dice or coins), and computer simulation would allow students to get an approximate solution for these original problems in an intuitive way without an excess of mathematical formalism.

Representations

Computers provide today new dynamic and graphical representation (as regards those present in the TCL history) that provide opportunity for student to explore the TCL, vary the different assumptions and observe how they affect the results. For example delMas, Garfield, and

Chance (1997), developed a simulation software and instructional and assessment materials to guide students' exploration and discovery of sampling distributions, where students can vary the type of distribution, and use non standard models. However, the reference to these kinds of tools is very scarce in the textbooks, which also reduce graphical representations with an overuse of algebraic formalism.

Procedures

Procedures in the books are almost reduced to algebraic computation or to computing probabilities from the tables of normal distribution. In our proposal we will make a wider use of tools, such as deducing sampling distribution from manipulative and computer simulations, studying the effect of n and of parameters in the initial distribution on the speed of convergence and the goodness of approximation, visualization sampling distributions for different sample sizes, etc.

Mathematical Objects and Properties

There is no discussion of the conditions for the TCL general validity, which may lead to an overgeneralization and to the belief the TCL is true, independently of the starting conditions. There is no study of the precision in the approximation, which may lead to generalization of results to small sample sizes.

Arguments

The main type of argument used in the textbooks is algebraic deductive proofs; other times the results is given without any justification. We recognise that, given the scarce time available for teaching it is not possible to use with these students the original procedures (e.g., inverting characteristics functions). However between a too formal deductive proof and lack of proving we suggest the possibility mentioned above of allowing students' exploration of problems and TCL properties by resort to dynamic software and simulation.

BASES FOR A DIDACTICAL PROPOSAL

As a result of our previous study we are currently developing a didactic proposal to teach the TCL to engineers. This proposal is restricted to independently identical distributed random variables and we mix traditional work in the classroom with simulations using *@risk Excel* facilities. The proposal is organised in 3 lessons, each of them around a main original problem:

1. *Approximating the Binomial distribution*. We start from the problem of investigating the reliability of a system made of a big number of individual components, each of them with the same probability of failure, which serves to introduce the approximation of Binomial distributions $B(n, p)$ for increasing values of n. An experiment with manipulative materials serves to deduce an intuitive approximation to the normal distribution, which is later justified by the teacher in the classroom. Once the normal approximation to the Binomial distribution and the continuity correction is accepted, students are given application problems, such as deducing if a production process is controlled, computing the sample size to carry out an experiment, or testing a hypothesis concerning a proportion.

2. *Distribution for the sum of random discrete variables*. After reflecting that the Binomial distribution is the sum of n identically distributed independent random variables, students are asked to investigate if the result (normality of sum) would apply for the sum of other random discrete variables. A laboratory activity with the help of software and Internet applets consists in investigating if the result stands for discrete uniform distributions and deducing the sampling distribution of sums and averages for samples of increasing size. After students accept normal distribution would provide an acceptable fit when n is large enough, the teacher presents a provisory formulation for TCL (for discrete random independent and identically distributed variables). Applications will include approximation of Poisson distribution in practical problems such as number of telephone calls in a telephone

distribution centre, consumers' choices of some products, or expected number of defaults in a production process.

3. *Distribution of the sum of random continuous variables*. In the last unit students are asked to investigate whether the TCL stands for continuous variables. The introductory situation is determining the total voltage in a condenser receiving inputs from many different electric lines. Applications involve Weibull and exponential distributions in reliability applications, exploration of weekly expense in a given factory, error in measurement and cost of mailing. Laboratory activities serve to investigate the generalizability of TCL to different types of continuous distribution. Finally, a general formulation of the theorem and the conditions for validity are introduced by the lecturer.

CONCLUSIONS

The CLT is a fundamental tool in statistical inference; however time availability and students previous knowledge make lecturers not to pay enough attention to its teaching. Even when a formal proof of some of the different formulation of this theorem is beyond the understanding of most undergraduate students in service courses, we believe that spending some few lessons in investigating the CLT meaning is worthwhile. This is today made possible by computer dynamic software with graphical and simulation capabilities, as well as by Internet applets.

The next stage in our research is experimenting this didactical proposal, assessing students understanding and informing the scientific community about its results. In this way we expect to contribute to better understanding of didactic of statistics at University.

ACKNOWLEDGEMENT

Research supported by the grants FQM126, Junta de Andalucía and SEJ2004-00789, MEC; Madrid.

REFERENCES

Alvarado, H. (2004). *Elementos del Significado del Teorema Central del Límite en Textos de Estadística para Ingenieros*. Memoria de Tercer Ciclo, Universidad de Granada.

Batanero, C. and Godino, J. D. (1998). Clarifying the meaning of mathematical objects as a priority area of research in mathematics education. In A. Sierpinska and J. Kilpatrick (Eds.), *Mathematics Education As a Research Domain: A Search for Identity*, (pp. 177-195). Dordrecht: Kluwer.

delMas, R., Garfield, J., and Chance, B. (1999). Exploring the role of computer simulations in developing understanding of sampling distributions. Paper presented at the Annual Meetings of the American Educational Research Association, Montreal, Quebec, April 18-23, http://www.tc.umn.edu/~delma001/stat_tools/aera1999/comparison.htm

delMas, R., Garfield, J., and Chance, B. (1997). Assessing the effects of a computer microworld on statistical reasoning. Paper presented at the Joint Meetings of the American Statistical Association, Anaheim, CA, August 10-14, http://www.tc.umn.edu/~delma001/stat_tools/asa_1997_paper/asa_paper.htm

Godino, J. D. (2002). Un enfoque ontológico y semiótico de la cognición matemática (An ontological and semiotic approach to mathematical cognition). *Recherches en Didactique des Mathématiques*, 22(2/3), 237-284.

Fisher, H. (2000). *The Central Limit Theorem from Laplace to Cauchy: Changes in Stochastic Objetives and In Analytical Methods*, http://mathsrv.ku-eichstaett.de/MGF/homes/didmath/seite/1850.pdf.

Méndez, H. (1991). *Understanding the Central Limit Theorem*. Thesis Doctoral. Universidad de California.

Mether, M. (2003). The history of the central limit theorem. *Sovelletun Matematiikan erikoistyöt*, 2, 108, http://www.sal.tkk.fi/Opinnot/Mat-2.108/pdf-files/emet03.pdf.