Bridging Theory: Activities Designed to Support the Grounding of Outcome-Based Combinatorial Analysis in Event-Based Intuitive Judgment—A Case Study

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Summary

Li, an 11 year old boy, participated in the implementation of a mixed-media design for the binomial that combines activities pertaining to theoretical probability (combinatorial analysis) and empirical probability (simulated experiments). This design was engineered to accommodate, corroborate, yet elaborate on students' heuristic inferences, and student reasoning was elicited through semi-structured clinical interviews. Applying a cultural–semiotic approach to the analysis of Li's case study, I discuss a universal pedagogical tradeoff articulated as tension between constructivist and sociocultural perspectives on mathematics education. Li fluctuates between two interpretations of a sample space: event-based attention grounded in intuitive perceptual judgment of a random generator yet oblivious to permutations; and outcome-based attention supporting normative mathematization yet initially unsynthesized with intuition. These apparently vying perspectives are reconciled, if problematically, when Li notices that the entire sample space indexes an expected distribution qualitatively aligned with his perceptual intuition. At a theoretical level, I argue, constructivist and sociocultural perspectives, too, can be reconciled, if problematically, by accepting that mathematical phenomena are phenomenologically akin to scientific phenomena and thus mathematical learning is an inductive process of synthesizing *(Schön, 1981) heuristic-based perceptual judgments and artifact-based mediated analytic procedures.*

Introduction

Leading education researchers of probabilistic cognition generally agree as to the pedagogical value of enabling students to explore the complementarity of two genres of investigative activities, 'theoretical' and 'empirical,' targeted at random generators: (a) combinatorial analysis as a procedure for determining expected outcome distributions in actual experiments with the generators; and (b) experimentation, typically computer simulated (Jones, Langrall, & Mooney, 2007). Yet how should this complementarity be facilitated? Namely, which media may best support such inquiry? What are the perceptual experiences, cognitive requisites, and semiotic contexts that optimize a connecting of the knowledge associated with each of these activities? What pedagogical approaches might best foster participant students' learning in such designs? How might normative mathematical concepts and procedures emerge from these activities?

Design solutions vary widely. Some researchers present students with computer-based models of familiar random generators, e.g., dice, and enable the students to alter properties of simulated experiments to explore the impact of these alterations on the distribution of actual outcomes (Drier, 2000; Konold, Harradine, & Kazak, 2007; Pratt, 2000). Other researchers have the learners themselves author computer programs, thus to "debug" their own misconceptions regarding randomness, sampling, and distribution (Abrahamson, Berland, Shapiro, Unterman, & Wilensky, 2006; Abrahamson & Wilensky, 2003; Papert, 1980; Wilensky, 1995). Yet other researchers engage students in competitive games in which chances have been pre-distributed unevenly, so that the students are motivated to discover the sample space as an explanatory mechanism of the "unfair" empirical results they encounter (Amit & Jan, 2007). Finally, some studies have examined the potential of classroom collaboration as a resource for understanding probability, with particular attention to aggregations, either of pooled sample-space outcomes from coordinated combinatorial analyses (Abrahamson, Janusz, & Wilensky, 2006; Eizenberg & Zaslavsky, 2003) or numerous actual outcomes from many individuals' experiments (Abrahamson & Wilensky, 2005; Kuhn, Hoppe, Lingnau, & Wichmann, 2006; Vahey, Enyedy, & Gifford, 2000).

This variety of approaches indicates that the research community has not yet converged on an optimal design solution for basic probability. However, because each designer inevitably weighs tradeoffs while considering design solutions (Edelson, 2002), such as between pedagogical objectives and assessment constraints, it is natural that different designers arrive at different solutions as well as at different emphases in evaluating learning outcomes. Accordingly, this paper does not attempt to arbiter among the designs, which each has its own merits. Rather, the paper discusses our own recent design solution and in particular the consequences of implementing this design, both for students' experiences and, through analyzing these experiences, for our accumulating understanding of mathematical learning, within probability content and possibly beyond. Namely, through explaining the rationale of our activities and closely analyzing a case study of a student participant, I will create context for commenting on the nature of learning and its implications for design.

In the studies cited above, participants' initial judgments were by-and-large incorrect yet the design successfully engaged the participants in activity-based reflection on these thwarted expectations as a means of promoting deep understanding. Indeed, contriving situations in which students experience cognitive conflict—*breakdown* (Heidegger, 1962), *disequilibrium* (Piaget, 1971), *expectation failure* (Schank, 1996), or *learning issues* (Abrahamson, 2004, 2006a, 2006b; Abrahamson & Wilensky, 2007; Fuson & Abrahamson, 2005)—is a widely used, if contested, design principle (Tirosh, Stavy, & Cohen, 1998)—a principle that appears to engender accommodation of action schemas core to Piagetian genetic epistemology and vital to creativity (Steiner, 2001; Wilson, 1998). Yet, the nature and timing of the designed conflict, I submit, may have consequences for learning, both of content and habits of mind.

That is, in some traditional designs the conflict students experience is between their own naïve intuition, which is empirically exposed as flatly incorrect, and the teacher's authoritative mathematics, which turns out to be superior for actual prediction. For example, students unsuccessfully guess actual experimental outcomes of a random generator then learn to conduct combinatorial analysis that successfully anticipates these outcomes. Emerging from such designs, mathematical knowledge is liable to appear to students as arbitrary, tautological, hovering in inter-referring limbo between impenetrable theory and incontestable empiricism (Abrahamson, 2007a). Worse, students may learn to discount their own judgment as irrelevant to their mathematical reasoning. And yet, a pedagogical caveat is that the learner be supported in seeing "how mathematics will reconstruct his/her intuition" (Borovcnik & Bentz, 1991). Perhaps, I posited, a different nature and timing of conflict would foster learning experiences better aligned with constructivist pedagogical philosophy. Furthermore, I wondered whether contexts could be created in which the conflict would be resolved not through rejection of intuitive suppositions in favor of validated procedures but as reconciliation of intuitive and normative knowledge.

Furthermore, in order to engender such reconciliation, an aim of my design became to reengineer the conflict such that students themselves generate the competing notions, each notion *with* mathematical merit. In sum, I attempted to recast students' learning experience not as substitution of intuitive judgment with normative mathematics but rather as a focused negotiation between two mathematically plausible views that both "make sense," yet may be based in tacitly differing formulations of the situation. Practically, I would need to create mathematical situations that elicit students' vying perspectives and facilitate the reconciliation of these perspectives. Moreover, in order to hone these negotiations, I wished to situate the conflict in actual mathematical objects and facilitate the conflict as a struggle—personal, interpersonal—to articulate what these objects are, within goal-oriented problem-solving activity contexts (see Borovcnik & Bentz, 1991, on the didactical generativity of conflicting constructions of the problems).

I thus searched for a context in which students' *initial* intuitive judgments would be *affirmed* as mathematically normative rather than disproved.[1] My working solution was to enable students to cast initial judgment with respect to anticipated empirical outcomes from experimenting with some random generator then recognize consonance between this judgment and properties of the generator's sample space. Under such settings, I projected, students' experience would change dramatically from being proven wrong to learning to argue why they are right. As such, the design was to foster learning trajectories in which students appropriate normative procedures as means of warranting their initial inference, using discursive forms in line with practices of argumentation and proof valued by mathematicians. Note that I did not intend to eschew cognitive conflict altogether—rather, the conflict was to be deferred to a later stage in the activity sequence. The design challenge was, thus, to:

(1) create a problem context wherein students' intuitive judgment of expected distributions of experimental outcomes is mathematically normative; and
(2) provide expressive media that enable students to support these intuitive inferences using newly encountered mathematical procedures that the students come to appreciate as rhetorical devices triangulating their intuition (Abrahamson, 2007b, 2008, under revision).

Thus normative mathematics would be presented not as a superior substitute to intuition but as a new *semiotic means of objectifying the presymbolic* intuition (Radford, 2003); a semiotic means that, however, would not quite enable students to *articulate* their perceptual intuition but, rather, would construct a complementary formulation of the source situation (Abrahamson & Cendak, 2006)—and herein lies the potential conflict—a new formulation that could become viably *synthesized* with the intuitive sense (Schön, 1981). Students' consequent "designed struggle" toward synthesis is the topic of this paper.

Below, I begin by presenting the specific design solution I am currently investigating (Abrahamson, under-revision; Abrahamson & Cendak, 2006).[2] Next, I discuss in depth a case study of a middle-school student engaged in a problem-solving activity based on the design. This case, which demonstrates a learning trajectory typically observed in the study, illuminates a pedagogical dilemma germane to any design program informed by the following theoretical dialectic: radical-constructivists insist on fostering students' self-invention of mathematical concepts (von Glasersfeld, 1987, 1990, 1992), whereas socioculturalists perceive mathematical learning is a process of acculturation into mathematical practice through engaging artifacts, e.g., procedures, that mediate standard routines, thus instrumenting the individual toward phenomena (Ernest, 1988; Vérillon & Rabardel, 1995; Vygotsky, 1978/1930). The pedagogical dilemma is manifest when students experience their in-coming intuition as disconnected from the introduced practice (Wilensky, 1997). On the one hand, we cannot wait for students to "reinvent the wheel," but on the other hand we do not

[1] Classical studies have demonstrated humans' apparently inherent difficulty in reasoning rationally with respect to probabilistic situations (Konold, 1989; Tversky & Kahneman, 1974), so that dissonance between the intuitive and the normative appears inevitable. Yet, as Gigerenzer (1998) argues, those traditional experimental contexts do not cater to humans' evolved schemas for operating effectively in natural situations pertaining to the mathematical study of probability (see also Gelman & Williams, 1998).

[2] The design builds on previous studies (Abrahamson, 2006c; Abrahamson, Janusz et al., 2006; Abrahamson & Wilensky, 2002) that, in turn, extended the *connected probability* project (Wilensky, 1993, 1995, 1997).

wish to impose solution procedures despotically. Thus, whether the new tools are introduced brusquely or surreptitiously, the pedagogical challenge is to make these tools meaningful. How, then, can we help students perceive the tools as extending their intuitive grasp?

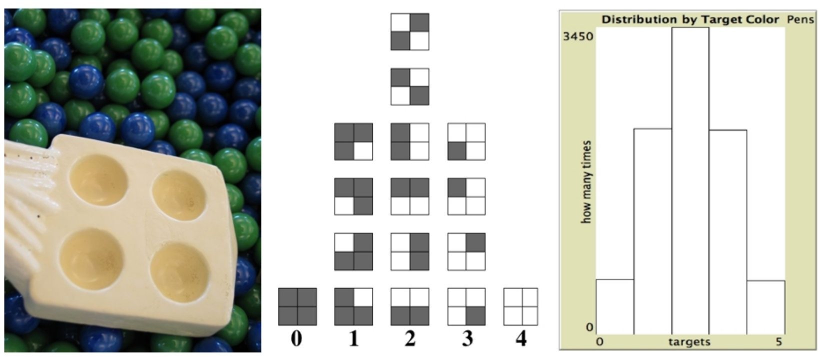
I will argue that design-researchers' constructivist-vs.-sociocultural theoretical conflict, reflected in students' intuition-vs.-artifacts emergent conflict, can be resolved, providing we recognize mathematical phenomena as ontologically equivalent to physical phenomena. Namely, both mathematical and physical phenomena are experiences that we apprehend and investigate, such that initial sensuous perception is explained through analytic observation. Just as the phenomenology of magnets is epistemologically distinct from the science of magnetism purporting to explain it, so the phenomenology of stochastics is epistemologically distinct from theoretical probability that provides analytic warrants for perceptual judgment (see also Liu & Thompson, 2002; Papert, 2000; Piaget & Inhelder, 1952).

What does this imply for the nature of mathematical understanding? Would students accept any arbitrary analytic process, just as long as it consistently supports their intuitive impressions? As I demonstrate, students will indeed appropriate mediated procedures, even if these initially appear as arbitrary, but only *if* and *when* the students perceive these procedures as consistent both in terms of following an internal pattern they can decipher and as producing inferences consonant with their intuition (Abrahamson, 2002), whether primary or secondary (Fischbein, 1987). As such, evidence of students' designed struggle to coordinate intuitive perceptual conviction and mediated analytic logicization epitomizes yet advances learning-sciences theoreticians' struggle to coordinate constructivist and sociocultural explanations of human learning (Cole & Wertsch, 1996). Thus, both binary sets of ostensibly conflicted approaches—students', scholars'—are reconciled in the praxis of mathematical learning. Namely, mathematical learning is the process of developing induction-based trust and fluency with mediated analytic calculations that *accommodate, corroborate, yet elaborate* (*a.c.e.*) intuitive judgments (see also Poincaré, 2003/1897; Salk, 1983; Schön, 1981; Whewell, 1989/1837; Wilson, 1998).

A Design for the Binomial

The design employed in this study included materials geared to support student activities pertaining both to empirical probability (a physical random generator and computer-based simulations of experiments with this generator) and theoretical activities (cards and crayons for creating the sample space of experiments with this generator). A single mathematical object, the *4-block*—a 2-by-2 grid of four squares, each of which can be either green or blue—crossed all the materials. Its empirical instantiation was in the form of a *marble scooper* (see Figure 1a), an utensil for drawing samples of (four) ordered marbles out of an urn-like tub containing hundreds of green and blue marbles of equal numbers, and computer simulations in which computational pictographs of 4-blocks stacked up in five *stalagmites* corresponding to the respective events of scooping exactly 0, 1, 2, 3, or 4 green marbles (see Figure 2). Other computer simulations yielded actual outcome distributions in normative histogram forms (see Figure 1c). A *combinations tower* (Figure 1b) was designed as a hybrid form built of the totality of possible experimental outcomes (the sample space) assembled in columns that mirrored the histogram structure.

The rationale of this hybridity was to create a bridging context between the theoretical and empirical inscriptions as a means to facilitate students' juxtaposition and coordination of these complementary constructs into a grounded understanding of the binomial. Specifically, these pre-probability students were first to intuit the expected distribution qualitatively—that the most likely scoop has exactly 2 green marbles, since neither color density is advantaged in the box, and that all-green or all-blue events are least likely—then blend this unformulated sense onto the tower they construct, so that its 1:4:6:4:1 columns are interpreted as indexing the intuitive distribution (see Hutchins, 2005, on material anchors for conceptual blends). The computer modules were aimed both to validate the intuitive-cum-analytic triangulated expectation of empirical distribution and offer a conceptual model for interpreting such distribution as a stochastic–multiplicative transformation on the sample space (Abrahamson & Cendak, 2006).



a. *b.* *c.*

Fig. 1. Theoretical and empirical embodiments of the 4-Block mathematical object: (a) The marble scooper; (b) the combinations tower; and (c) an actual experimental outcome distribution produced by a computer-based simulation. To interact with applets of the computer-based simulations, visit edrl.berkeley.edu/design.shtml.

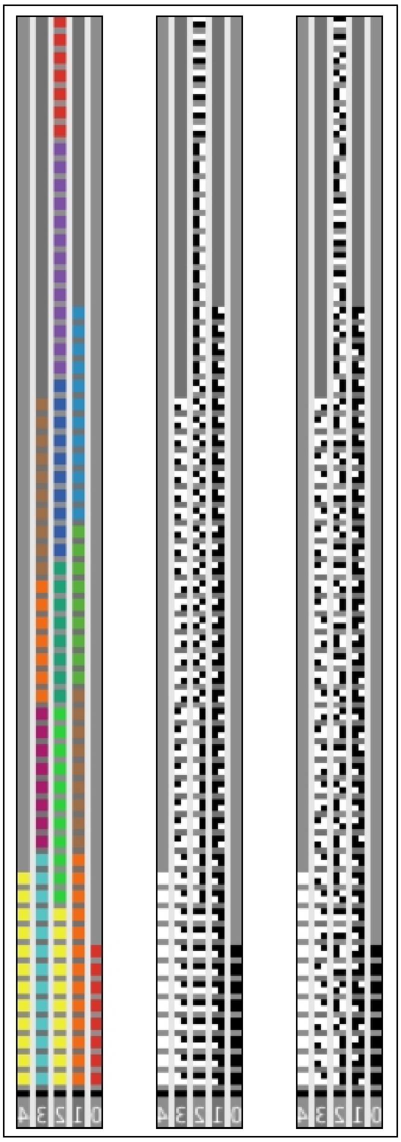


Fig. 2. "Stalagmite" outcome distribution can be re-ordered and/or colored within events by outcome categories.

Methods

The data used for this study were collected as part of a larger project, *Seeing Chance* (Abrahamson, PI), conducted in the *design-based research approach* (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Collins, 1992)—a *conjecture* driven research paradigm (Confrey, 1998)—in which researchers investigate the plausibility of some teaching or learning *mechanism* (Confrey, 2005), such as grounding students' analytic understanding of stochastic distribution in their intuitive expectation of frequency through engaging students in activities supported by materials (artifacts, activities, facilitations) designed to achieve the pedagogical goal. Initially, design-based researchers' instantiate their conjecture in the form of a first-take set of pedagogical materials. Then, through a sequence of empirical studies, the researchers iteratively tune the design in response to consistent patterns in participant students' observed behaviors. Reciprocally, by reflecting on the nature of these tunings, the researchers define and refine their theoretical models, often developing new constructs that may obtain beyond the narrow context of the specific content ("humble theories," Cobb et al., 2003; 'ontological innovations,' diSessa & Cobb, 2004).

Li was one of 28 middle-school students (9-12 yo) we engaged each in a 1-hour one-to-one clinical interview (diSessa, 2007; Ginsburg, 1997). All sessions were video recorded for subsequent analysis. An interview protocol structured the following activity sequence (using age-appropriate mathematical vocabulary). Participants: (a) predicted the actual outcome distribution in a hypothetical stochastic experiment conducted with the marbles box and scooper; (b) built the sample space of the experiment; and (c) operated the computer-based simulated experiments.

Throughout, we used prepared questions and extemporized follow-up discourse to elicit students' observations and reasoning. In line with the design framework, *learning axes and bridging tools* (Abrahamson, 2006a; Abrahamson & Wilensky, 2007), the interviewer steered the student toward experiencing a succession of cognitive conflicts pertaining to conceptual issues identified as key to the domain. These conflicts emerged from competing interpretations of objects intentionally designed to be ambiguous. For example, an outcome in the sample space, e.g., the 4-block with a green square in its bottom-left corner, might under certain contextual circumstances be interpreted by a student as just that, a single outcome, yet under other circumstances she might refer to the same object as signifying *any* of the 4 outcomes with exactly one green square. Having students acknowledge this issue and consider its procedural ramifications is potentially crucial for their learning to discriminate between combinations and permutations, a requisite distinction for understanding how combinatorial analysis predicts frequencies.

Whereas in previous studies we have taken broader analytic scopes of these data (Abrahamson, 2007c; Abrahamson & Cendak, 2006), the current paper focuses on a single case study so as to discuss in detail relations between problems, intuition, and media, with implications for an integrated cognitive and sociocultural theory of mathematical learning. Li, our case study participant, is a 6th grade (11'4 yo) student evaluated by his mathematics teachers to be in the top third of his class with respect to his achievement and participation (see Abrahamson, Bryant, Howison, & Relaford-Doyle, 2008, for case studies of students of other levels). I selected Li because his behavior was typical of his age/achievement group, yet he was particularly articulate in expressing his reasoning. A 3.5 min. video excerpt featuring the culmination of Li's learning process serves as the focus of our discussion in the next section.[3]

Our research team collaborated in intensive/extensive microgenetic qualitative analysis (Schoenfeld, Smith, & Arcavi, 1991) of the Li video excerpt in attempt to build a coherent explanation of his behavior in light of his entire interview as well as our whole corpus of data. We approached the dyad's utterances, gestures, and inscriptions as multimodal, multimedia, multi-semiotic-system goal-oriented expressive acts, in which objects' emergent meanings are negotiated, often tacitly, by strategically making salient their perceptual features and relations (Arzarello, Robutti, & Bazzini, 2005; Bartolini Bussi & Boni, 2003; Cobb & Bauersfeld, 1995; Lemke, 1998; Radford, 2003; Roth & Welzel, 2001; Stevens & Hall, 1998).

By way of introducing our inquiry into the focus video clip from the Li interview, I begin, below, by describing a general finding (Abrahamson & Cendak, 2006) that greatly challenged us yet ultimately led to our current hypotheses, which I then elaborate through attempting to explain the Li clip.

[3] Please see the video excerpt from the interview at http://edrl.berkeley.edu/conferences/icme/Abrahamson_ICME-2008-LF.mov

Results and Discussion

General Findings: The Baffling Case of Students' Inadvertent Metonymic Treatment of Outcomes

After students had completed the construction and assembly of the sample space, such that all 16 outcomes were arranged in the form of the combinations tower, the interviewer would lift two cards, a card from the 3-green (3g) column and the single 4-green (4g) card, and ask the participant whether these two outcomes are equally likely to be drawn (scooped) from the marbles-box random generator or whether one of them is more likely than the other (see Figure 3, next page).

By and large, students would claim that the 3g card is more likely. This, despite our taking measures to ensure that the interviewer and student were referring to the same physical object. For example, the interviewer would repeat that s/he is referring to the particular 3g card, not the group, and the student would concur. Similar results were obtained for other between-column card pairs. When asked a similar question regarding a pair of *within*-column outcomes, however, students would state that the outcomes were equiprobable. The dyad would then discuss the between-column issue until the interviewer was satisfied that the student stably differentiated between an outcome *qua* outcome and an outcome *qua* event. These negotiations lasted between 1 and 12 minutes, with the older and higher-achieving students curiously taking the longest to stabilize.

We have put forth the construct *inadvertent metonymy* as a description (but not yet an explanation) of student response (Abrahamson et al., 2008). By 'metonymy' we mean that the student referred to the single 3g card as signifying the entire event class of all four possible permutations in the column. By 'inadvertent' we qualify students' semiotic act by acknowledging that the metonymy was unintentional. With the construct of 'inadvertent metonymy,' we aimed to:

(a) highlight that the dyad agreed on the referent but disagreed on its sense—we thus shifted our analysis from attending exclusively to semantic and syntactic features of the discursive act to a broader view encompassing semiotic and pragmatic dimensions of the situation and deconstructing these pivotal interpersonal dimensions so as to deepen our understanding of students' individual cognition (see also Borovcnik & Bentz, 1991);
(b) suggest that the dyad's disagreement had been tacit—it was exposed only through interaction and, reflexively, that the researchers understood the confusion only through collaborative analysis;
(c) respect both dyad member views as viable, given the individuals' respective mathematical knowledge at that time; and, ultimately
(d) hone subsequent data analysis by focusing on the *meaning* of agreed referents and asking why the students were inclined to see outcomes as events.

In sum, we recognize that 'inadvertent metonymy' is a figment of the researchers' epistemology—an ontological innovation that helped us make a *first* pass at the baffling data by situating the students' rhetorical act within our own semiosis, yet a first pass that did not yet furnish a viable explanatory model for the students' reasoning. In the remainder of this section I will share more recent insights that our proposed interim construct 'inadvertent metonymy' has engendered.

I begin by reiterating that the mathematically informed researcher brings to bear well articulated frames that the student has yet to develop. Namely, the researcher clearly differentiates between the respective functionalities of outcomes and events within the combinatorial-analysis procedure: plausibly, in some contexts the researcher, too, might be inclined to refer to an outcome metonymically so as to mean the entire event. Yet when the property of likelihood is at stake, permutations must be enumerated as singular elements, such that the metonymical meaning of the card must give way to regarding the card as a singular entity within the sample space. The researcher, thus, would probably refrain from referring to a card at once both metonymically and singularly, because the researcher would recognize the rhetorical confusion such simultaneous conflicting signification might engender.

The students, however, initially lack conceptual or logical structures for moving adroitly between an event- and outcome-oriented views of a card – they have yet to notice and articulate features of the card associated with each view, e.g., one student used "color-wise" to mean a simple count that disregards the particular arrangement of green and blue squares in the 4-block and "place-wise" to mean attention also to the configuration of these colored squares. Yet I wish to emphasize that building mathematically normative conceptual structures through initially differentiating perceptual attributes of objects is entirely contingent on the perceived goals of an activity. These students do not yet know that attention to permutations is crucially conducive of combinatorial analysis; in fact, they do not at all know they are engaging in what we call combinatorial analysis!

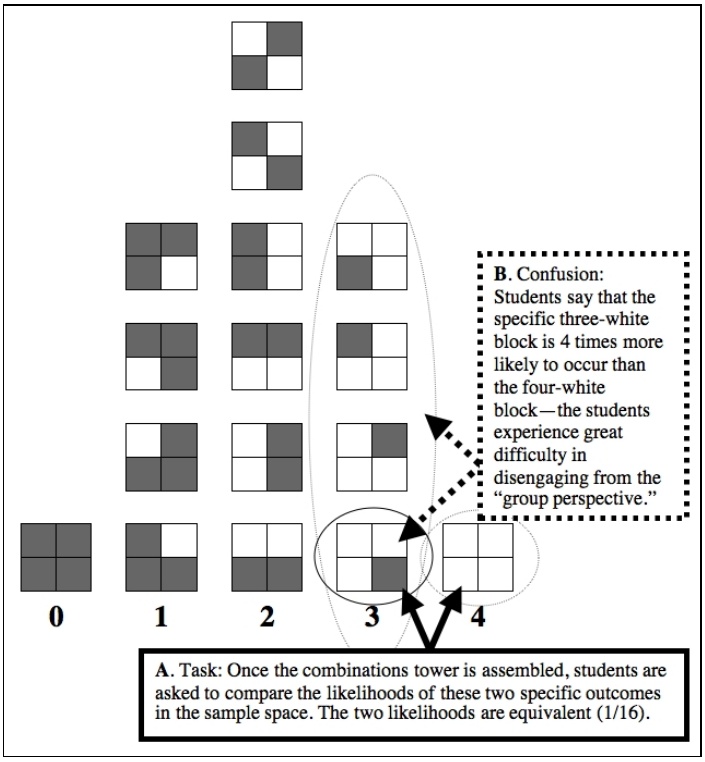


Fig. 3. Students' event-vs.-outcome confusion.

That said, the goal of an activity may emerge telescopically through negotiating discursive contingencies that give rise to perceptual–conceptual categories appropriated as semiotic tools. Initially, however, the students gloss over events and outcomes as functionally synonymous and therefore non-conflicting co-occurrences within a single discursive act.

As noted, our data-analysis inquiry shifted toward scrutinizing the microgenesis of students' event-based perception of the sample space. Within the brief history of the interview, we therefore asked, what activities gave rise to students' event-based orientation toward the sample space? As the narrative of the Li case study will reveal, we have implicated the available media—specifically the interaction between students' contextual intuitions grounded in the marbles box and the media made available for subsequent combinatorial analysis—as explanatory of students' rhetorical acts. Students' persistent event-based seeing of individual outcomes will be explained as logical and argued as transitional toward normative understanding. In so doing, I submit that the available media were challenging yet instrumental in Li's conceptual development—the media bootstrapped insight. In particular, I will interpret Li's difficulty as *rupture* caused by the ontological duplicity of the *semiotic means of objectification* (Radford, 2003) made available for him to reify his presymbolic psychological objects. Thus, the Li case study will problematize cultural tools as double-edged swords, from a constructivist perspective: The tools are to accommodate students' intuitions, yet in order to *amplify* (Norman, 1991) those intuitions, these tools—perhaps necessarily—incorporate "frictive" elements that may initially jar with this intuition.

Analysis of the Li Case Study: Ontological Imperialism in Available Semiotic Means of Objectification

At the beginning of the interview, Li was asked to predict "what we will get" (the *probable*) when we scoop from the marbles box. He gazed at the box and concluded that '2g–2b' would be the most common draw. Subsequently, he was handed the stack of empty 4-block cards and asked to show "what we could get" (the *possible*). He builds a set of five events (see Figure 4, on the right, the bottom row) and states that he has completed the task. Namely, Li is content that the five cards exhaust the sample space, e.g., Li sees the card that has a single light-colored square in the top-right corner as signifying the 1g *event*, and makes no allusions to the other three singular permutations on this event (see, in Figure 4, three additional grayed-out permutations above that card).

The interviewer, however, perceives this same card as objectifying one particular outcome out of the set of four possible permutations. That is, the card is an ambiguous semiotic object, and the dyad tacitly disagrees over its meaning—whether it is an event or an outcome. Li does not initially realize the implications of his representation, i.e., that he has unwittingly been subjected to *ontological imperialism* (Bamberger & diSessa, 2003) through interacting with the media put at his disposal as expressive means. Namely, Li was seeing events as the 'things' in the experiment, yet the media implicitly coerced him to create outcomes as the privileged elemental unit.

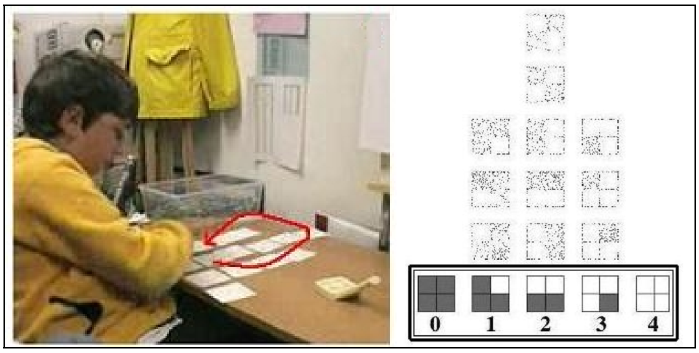


Fig. 4: "These don't really matter" (+ gesture): the sample space, as the student initially sees it.

Indeed, as the ensuing conversation reveals, Li sees the cards as events, not outcomes, because his communicative goal is to objectify the presymbolic notion that corresponds to events—not outcomes—as framed by the initial task he is still entertaining. That is, Li's presymbolic categories were order-less combinations, yet the cards' structural properties added specificity, rendering order-less combinations necessarily ordered permutations. Li is unaware that by virtue of using the available media, he has represented his intuitive judgment such that a skilled user of the medium reads into these representations meanings—new layers of signification—that Li had not intended or even harbored.

Namely, these inadvertent cues are "visible" only to practitioners in this disciplinary domain (see Goodwin, 1994, on 'professional vision'), for whom order counts in the context of conducting combinatorial analysis. For Li to attend to order, the dyad must first acknowledge that they are differing in their *orientations of view* (Stevens & Hall, 1998) toward the cards, objectify and name these orientations, and then negotiate whether one of the orientations is more advantageous toward achieving the common goal. At this point, though, Li is oblivious to the goal of combinatorial analysis, and so the process of building the five cards appears to him as no more than an opportunity to reiterate his earlier statement, using semiotic objects conducive to deixis (in particular, pointing).

In sum, Li does not initially know that by attending to order he can create a set of objects, the sample space, that would collectively index his presymbolic notion. The complete mathematical pertinence of the combinatorial-analysis procedure is thus temporarily suspended—the procedure will be *instrumentalized* (Vérillon & Rabardel, 1995) only once Li sees the sample space (the product of this procedure) in its entirety as indexing his presymbolic notion; only once Li *objectifies* (Radford, 2003) the combinations tower as resonant with his intuitive judgment; only once he *blends* (Fauconnier & Turner, 2002; Hutchins, 2005) the unarticulated sense of frequency, emanating from the marbles-box context, into the 16 cards; only once he *synthesizes* the intuitive and formal (Schön, 1981). Thus the goal of the activity of coloring the cards is enfolded into their "hidden" property of order, waiting to emerge when the task is completed.[4]

Yet ontological imperialism predicts consequences more dire than communication breakdowns between a student and a teacher. Namely, the student may reflexively come to see the world through the lenses of the imposed representation and in so doing may lose unarticulated notions vital for grounding these mediated representations within the original situated experiences (Bamberger & diSessa, 2003). As we shall see, Li is indeed affected reflexively by the emerging meanings of the mathematical objects he created—Li enters a transitional phase of generative confusion, as follows.

Gazing at the five cards he had just created, Li retracts his initial intuitive judgment, stating that the five events are equally likely to occur! It is as though Li has unburdened/distributed the onus of information onto the cards and now, gazing afresh, tacitly trusts the cards—the stationary stationery—to re-evoke his presymbolic notion, which he had ostensibly inscribed. Yet just as the 4-block structure of the empty cards compelled an ordered display of an order-less notion, so now the individual cards' lack of affordance for marking frequency engendered the apprehension of a flat distribution—by giving Li more representational media than he could manage, he has temporarily mangled the message. Thus, crucial aspects of a presymbolic notion are attenuated once the notion is mirrored through a person's constrained fluency with a representational system, breeding conflict between the intuitive and the inscribed.

The interviewer's objective becomes to enable Li to regain his presymbolic sense of the distribution and sustain it by appropriating semiotic means of objectification in keeping with mathematical convention and validity. He asks Li to create the permutations on the five combinations. Li builds the remaining 11 cards yet protests that the permutations are not pertinent to the analysis, because the initial question was about events, not outcomes (see Figure 4, on the left). Li's contention, paraphrased, is, "If you want me to predict the frequency of *combinations*, why should I care about all these *permutations*?" Then, upon further scrutiny, Li sees the middle column as implying a 6/16 chance for that 2g–2b event. It is as though Li has "shifted gears" and is now building an analytic argument in a mental space that temporarily cordons off the event-based intuition. Yet as his gaze wanders to other parts of the tower, specifically to the 4g single-outcome event, Li reasserts his flat-distribution hypothesis. In mid sentences, he keeps alternating.

The interviewer prompts Li to re-consider the marbles box, thus re-evoking for Li the activity's initial context, then, gesturing at the combinations tower with vertical strokes, the interviewer highlights the columns while reiterating Li's contradicting values for the flat distribution. Li experiences resonance between the sample space and his initial intuition: he sees and encodes the columns' relative heights or counts as indexing the anticipated plurality of 2g–2b (see the heuristic in Stavy & Tirosh, 1996). Thus, a *consilience of inductions* (Whewell, 1989/1837)—(a) an anticipation of outcome distribution based in perceptual judgment of the marbles box that is experienced as consonant with (b) a heuristical reading of the combinations tower—placates Li's reluctance: now that the new representation "feels" right, Li is willing to appropriate combinatorial analysis retroactively as instrumental in predicting distributions.

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[4] Given viable semiotic means of elaborating on the five cards (the objects) such that they do connote the presymbolic qualitative distribution (the properties), Li may likely appropriate those means. One could imagine a variety of such means, e.g., using a red marker to indicate upon the five cards the "intensity" of their respective felt likelihoods. Conversely, one could give Li five cups, in which he would place actual marbles that would move around order-less-ly, e.g., 3g–1b. Yet only combinatorial analysis offers a semiotic means in line with the cultural practice of mathematical argumentation—the combinations tower's five columns inscribe the felt frequencies as the events' vertical projections and, so doing, render the qualitative notion quantitatively explicit.

Conclusion

Human intuition pertaining to long-run consequences of operating random generators is event based, whereas the mathematical analysis of these processes—whether piecemeal combinatorial analysis or pre-captured in the binomial function—is outcome based. Thus, people who judge a HTTH 4-coin flip as more likely than the equiprobable HHHH flip are not in error. Rather, they are applying sound intuition to formulate their own practical interpretation of the referents, i.e., they are counting outcomes yet ignoring their order, thus successfully comparing the likelihoods of the order-less events "2H–2T" and "4H" (see Borovcnik & Bentz, 1991, for similar claims regarding students' legitimate reconstruction of problems).

The *pedagogical* design problem this paper has grappled with is how to enable learners to sustain their event-based intuition as they learn to appreciate the rationale and utility of outcome-based analysis, in which binomial events each unfold as classes of equiprobable permutations. The *theoretical* problem examined through the case study pertained to the nature of human artifact-based learning and its implications for a coordination of constructivist and sociocultural theory of learning as well as for effective design frameworks.

Li's passage from intuition to inscription, though ostensibly complete, was a bumpy passage that may have left scars on the traveler… Though a logical, procedural, perceptual string does connect Li's presymbolic intuition to the interpreted sample space, this string is quite threadbare, perhaps leaving Li in a state that Wilensky (1997) calls *epistemological anxiety*, i.e., knowing that you know how to work something but not knowing why it works the way it works. A radical implication of the above would be that students are to be given all the leisure, latitude, and learning materials needed to create their own idiosyncratic representations of their intuitive notions—to create their personal paths *from intuition to inscription* (Abrahamson, 2007b), *from phenomenology to semiosis* (Abrahamson, 2008). Yet, the necessity of shared semiosis (not to mention assessment-based exigencies of school contexts) appears to deem such a position impractical. Even radical constructivist pedagogical practice is not about leaving children as "free range" agents in the learning environment (von Glasersfeld, 1992). The question evoked by the Li data, then, is how to foster students' reflective negotiation between their presymbolic notions and target representational and procedural systems, e.g., how to ethically establish a sample space as the *epistemic form* for predicting random distribution and combinatorial analysis as the *epistemic game* (Collins & Ferguson, 1993) one must play to construct this form.

To begin answering this question, I reinterpreted the constructivist caveat of departing from *where the students are* by examining what the students are *trying to express* as they engage in problem solving, given the available media. This cultural–semiotic approach to cognition (Arzarello et al., 2005; Radford, 2003, 2006, 2008), I find, is uniquely suited for deconstructing mathematical learning in multimodal multimedia contexts (Abrahamson, under revision). Therein, situated mathematical problem solving transpires as expressive acts formulated reflexively in and through available media, whether material, epistemic, or blends thereof. The teacher's charge is to highlight the media's implicit semiotic affordances, thus inviting students to reflect on emergent ambiguities in the situations and ultimately reconcile these ambiguities as grounded syntheses of intuitions and normative practice.

Equipped with these new insights, we may dissolve apparent tension between constructivist and sociocultural theory—a theoretical dissolution that, in turn, may breed design solutions for guiding students' safe passages along disciplinary trajectories from intuition to inscription. Thus, bridging tools may support the consilience of theoretical and empirical perspectives both in mathematical learning and in research onto this learning.[5]

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