TEACHERS' CONCEPTIONS AND CONSTRUCTIONS OF PEDAGOGICAL REPRESENTATIONS IN TEACHING ARITHMETIC AVERAGE ®

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This study examined twelve inexperienced and eleven experienced teachers' constructions of and conceptions about pedagogical representations for teaching arithmetic average. The teachers were asked to generate appropriate pedagogical representations as well as predict and evaluate the uses of different representations for solving problems involving the arithmetic average. The experienced teachers were able to predict a variety of representations as well as errors that are recognized as common among middle-school students, while the inexperienced teachers used algebraic representations almost exclusively. Additionally, the inexperienced teachers tended to value algebraic solutions over guess-and-check or visual drawing solutions, more so than did the experienced teachers. However, the differences in the experienced and inexperienced teachers' abilities to predict and evaluate the use of different representations were not clearly evident in their generation of pedagogical representations in a lesson plan context.

"Representations" are both an inherent part of mathematics and an instructional aid for making sense of mathematics (Ball, 1993; Dufour-Janvier, Bednarz & Belanger, 1987; NCTM, 2000). In mathematics, a representation must necessarily be used to express any mathematical object, statement, concept, or theorem (Dreyfus & Eisenberg, 1996). "Virtually all of mathematics concerns the representation of ideas, structures, or information in ways that permit powerful problem solving and manipulation of information." (Putnam, Lampert, & Peterson, 1990, p. 68) Pedagogically, representations are used by teachers and students in their classroom as carriers of knowledge and as thinking tools to explain a concept, a relationship, a connection, or a problem-solving process (Shulman, 1986). Adequate representations play important roles in students' learning and understanding of mathematics (Hiebert & Carpenter, 1992). Therefore, it is important to select pedagogically sound representations to foster students' understanding of mathematics.

When students understand mathematics, they are able to use representations to express mathematical ideas and problems, and are also able to move fluently within and between representations (Hiebert & Carpenter, 1992; Putnam et al., 1990; Dreyfus & Eisenberg, 1996). Several researchers found that teaching which takes into account pedagogical considerations of problem representations could facilitate students' problem solving and understanding (Greeno, 1987). Pedagogical representations are effective in classroom instruction if they are either known by students or are easily knowable. Although there is no universal agreement about what constitutes "pedagogically sound representation" in mathematics teaching, no one questions the idea that the construction of sound pedagogical representations is influenced by teachers' beliefs, conceptions, and knowledge of mathematics as well as by their teaching experience (e.g., Thompson, 1992). For this reason, this study examined eleven experienced and twelve inexperienced math teachers' conceptions and construction of pedagogical representations in their teaching of the concept of arithmetic average.

This concept provides an interesting context in which to understand teachers' conceptions and construction of pedagogical representations. The arithmetic average is both a central concept in statistics as well as a computational algorithm (Strauss & Bichler, 1988; Mokros & Russell, 1995; Watson & Moritz, 2000). It is a statistic (the mean) that is used to describe and make sense of a data set, and it is a tool, used in conjunction with the standard deviation, for summarizing a data set and comparing data sets. It is a computational algorithm for finding an average by adding the values to be averaged and dividing the sum by the number of values that were summed or by "evening-out" several quantities. Understanding the arithmetic average requires an in-depth understanding of both the computational algorithm and the statistical relationships of the concept (Cai, 2000). In particular, understanding the average involves (1) having procedural knowledge of the algorithm, (2) a conceptual understanding of the algorithm, and (3) a conceptual understanding of the concept as a statistic (the mean) to describe and make sense of a data set and to compare data sets. With a conceptual understanding of the averaging algorithm, students should be able to correctly apply it to solve problems in various contexts. With a conceptual understanding of the arithmetic average as a statistic (the mean), students should be able to make decisions as to when and how the mean can be used to describe and make sense of a data set or compare data sets.

METHODS

Selection of Teachers. Twelve inexperienced teachers (denoted as N1, N2,..., N12) and eleven experienced teachers (denoted as E1, E2,..., E11) participated in the study. Nine of the twelve inexperienced teachers were chosen from among a cohort of professionals who were pursuing teaching as a second career. Their original college majors were in physics, economics, computer science, business, communication, engineering, or mathematics, and each held a B.S., M.A., or Ph.D. degree. They had worked in other professions for a number of years and had decided to return to school for a teaching certification. These nine inexperienced teachers had no prior teaching experience in a formal school setting, except some tutoring experience. Another of the inexperienced teachers had earned a B.A. in mathematics some years ago, and had returned to school for a teaching certification after her children were grown. The remaining two inexperienced teachers recently graduated from colleges with degrees in mathematics and psychology, and have had one to two years of teaching experience in mathematics.

The eleven experienced teachers were chosen among 6th- and 7th-grade teachers who have taken leadership roles in their schools and/or school districts. In particular, all the experienced teachers have lead workshops or made presentations at regional or national math education conferences. All but one have been teaching for more than ten years, with a mean of 14.4 years and a range of five to twenty-three years. The teacher with only five years experience was chosen on the basis that she had been recognized as the outstanding first-year teacher in her state. Each holds a bachelor's degree in education; on top of that, two have master's degrees in education and one has earned a Ph.D. in math education, and four others are taking graduate-level courses in mathematics education. One teacher was recognized as district teacher of the year, while another received the Presidential Award for Teaching Excellence in Math and Science, which is among the highest honors in the teaching profession.

Data Collection. Data for this study was collected in three ways: (1) The teachers were asked to write an introductory lesson plan on arithmetic average, (2) the teachers were asked to think of possible ways 6th- or 7th-grade students might solve each of five problems involving arithmetic mean, and (3) the teachers were given six student responses to two of these five problems and were asked to evaluate the responses using a general five-point scoring rubric (using ratings of 0-4). The lesson plans and interview transcripts were analyzed to examine the teachers' conceptions and constructions of pedagogical representations. These different sources of data contributed information about representations from three aspects: (1) generating pedagogical representations for classroom instruction. (2) knowing students' representations and strategies in problem solving, and (3) evaluating students' representations and solution strategies.

RESULTS

Knowing Students' Representations and Strategies. All experienced and inexperienced teachers were able to solve each of the five problems. However, there were remarkable differences between the experienced and inexperienced teachers when they were asked to think about possible ways their students might use to solve each of the five problems. First, eight experienced teachers, but only two inexperienced teachers, provided multiple solutions to at least one of the problems. Second, while the experienced teachers very frequently predicted that their students would use either guess-and-check or drawing strategies to solve the problems, only one inexperienced teacher, Teacher N2, used any methods other than an equation-solving approach with strictly mathematical notations to solve each of the five problems. Figure 1 shows the two ways N2 used to solve the Hats Average Problem (*Angela is selling hats for the Mathematics Club [a picture is given showing that 9 hats, 3 hats, and 6 hats were sold in Weeks 1, 2, and 3, respectively]. How many hats must Angela sell in Week 4 so that the average number of hats sold is 7?*):

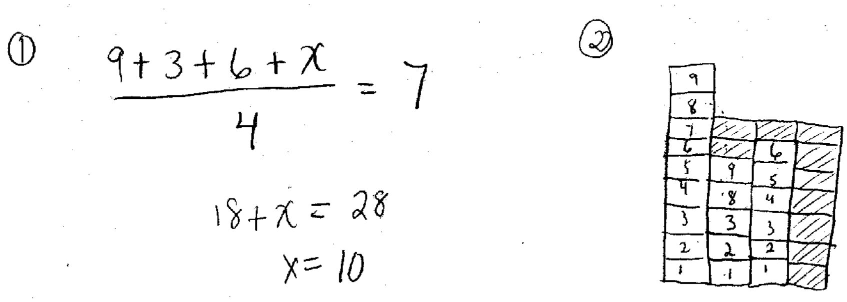

Figure 1. Two Solutions from Teacher N2

The third difference between experienced and inexperienced teachers was related to identifying student misconceptions. None of the inexperienced teachers mentioned students' misconceptions, but the experienced teachers consistently did. Although both sets of teachers were asked to think about ways their students might use to solve the problems, the inexperienced teachers drew mainly on their own approaches to solving the problems, while the experienced teachers were able to think what students are likely to think about. Experienced teachers predicted two major errors which students might commit. One is the incorrect use the averaging algorithm and the other is unjustified manipulating with numbers in a problem. For example, for the Elevator Average Problem (*There are ten people in an elevator, four women and six men. The average weight of the women is 120, and the average weight of the men is 180 pounds. What is the average of the weights of the ten people in the elevator*?), Teacher E1 predicted:

> *... I think that they might take 120 plus 180 and divide it by 10. Now that there's 10 people all together, this is an average, this is an average, so I add them together and divide by 10. I think some of them would do that. I don't know if some of them, we've been working so much with mean, median, and range, some of them might take the 120 plus 180 and divide by two. We've been talking about median when you get two in the middle. I don't know how well they'd do with this one. ...Because I just think that they like, you know whatever numbers are right there in front of them, those are the two numbers they're going to use*.

Evaluating Students' Solution Representations and Strategies. Table 1 shows the teachers' evaluations of the six student responses to the two problems involving arithmetic mean (See Appendix A). For Responses A, C, and D, both experienced and inexperienced teachers scored them highly. For Response B, the experienced teachers scored it slightly higher than did the inexperienced teachers. For the majority of the experienced and inexperienced teachers, using drawing to even out the number of hats sold in four weeks was a viable solution strategy; thus Response B should have been scored as 4. However, some experienced and inexperienced teachers did not score Response B 4 points because they felt that the drawing strategy to even out the number of hats sold in four weeks was not as good as the strategies used in Responses A or C.

For both experienced and inexperienced teachers, Responses E and F were the most challenging ones to evaluate. All the experienced teachers scored Response E 3 or 4 points, but the inexperienced teachers' scores ranged from 1 point to 4 points. Several inexperienced teachers (N2, N 9, and N10) scored Response E 1 or 2 points because they did not understand the thinking involved in the response. There are quite large variations in both experienced and inexperienced teachers' scores of Response F. The variation for the experienced teachers is bigger than that for the inexperienced teachers, although the large variation in both groups is due to differing views about accepting an estimate as an answer. Some teachers gave it 2 or 3 points because they believed that students understood the problem and used some properties of arithmetic mean to find an answer, with Teacher E10 commenting: "They could do better because of the information given to them, but I guess without using all the information they've attacked the problem well. This student proved an understanding of the concept." Other teachers gave it 0 points or 1 point because they did not feel an estimate constituted an acceptable answer when students had enough information to find an accurate answer, as Teacher E7 articulated: "They've got everything to figure out the problem, but they seem do not care and don't use them. Regardless how good the estimate is, it is just a wrong answer."

Table 1
Teachers' Scores of Students' Responses

Experienced	Response A	Response B	Response C	Response D	Response E	Response F
E1	4	4	4	4	3	0
E2	4	4	4	4	3	2
E3	4	3	4	4	4	3
E4	4	3	4	4	4	3
E5	4	4	4	4	3	2
E6	4	4	4	4	4	2
E7	4	4	4	4	3	2
E8	4	4	3	4	4	1
E9	4	4	4	4	4	3
E10	3	4	3	3	4	3
E11	4	4	4	4	4	3
Inexperienced	**Response A**	**Response B**	**Response C**	**Response D**	**Response E**	**Response F**
N1	4	4	4	4	4	2
N2	4	4	4	4	1	1
N3	4	3	4	4	4	2
N4	4	4	4	4	3	1
N5	4	4	4	4	3	1
N6	4	2	4	4	3	2
N7	4	3	4	4	4	2
N8	4	4	4	4	4	2
N9	4	3	3	4	2	0
N10	4	4	3	4	1	2
N11	4	3	4	3	4	2
N12	4	4	3	4	3	1

Construction of Pedagogical Representations in Lesson Plans. While the results of the two interview tasks revealed a greater richness in the experienced teachers' conceptions of pedagogical representations as compared to those of the inexperienced teachers, this greater richness was not always evident in comparing the lesson plans of the two groups. In many aspects the lesson plans were quite similar, and for aspects in which they differed significantly, the lesson plans of the experienced teachers tended to be less exemplary. Perhaps the most striking difference, given the experienced teachers' greater ability to predict both student responses as well as misconceptions in the first interview task, was the inclusion of expected student responses in the lesson plans. Only two of the eleven experienced teachers included expected student responses in their lesson plans, as compared to seven of the twelve inexperienced teachers. This difference might be due to the fact that experienced teachers used almost exclusively the "outline-worksheet lesson plan format."

In several other aspects, the two groups were fairly similar. For example, all the teachers used real-life situations and/or physical models to introduce arithmetic average. In particular, nine of the eleven experienced teachers and six of the twelve inexperienced teachers used manipulatives or actual physical measurement activities (e.g., measuring each student's height) to introduce the concept. The manipulatives and measurement activities were used in one of two ways by the teachers, in similar proportions in each group: One way was to generate data (E1, E2, E3, E8, N6, N7). For example, in Teacher E1's lesson, students were given some chips and asked to count them. In Teacher E8's lesson, students were asked to measure their heights and lengths of arms, and then see if "each student is average."

The other way was to model the process of finding the mean (E4, E5, E6, E9, E11, N5, N8, N11, N12). For example, Teacher E6 modeled the averaging process by having students move the cubes from one tower to another until the towers were even. Teacher E6 focused on the process of evening out the cubes. In fact, the objective of Teacher E6's lesson was "to have the students learn that averaging is similar to the process of leveling 2 columns (or more) of cubes." However, only a few teachers (E4, E6, N8, N12) who used manipulatives to model the process of finding a mean made a connection between the evening-out process with the written form of the averaging algorithm (Cai, Moyer, & Grochowski, 1999). For example, Teacher E4 used manipulatives to model the process of finding a mean, but she also used the evening-out process to help students discover the algorithm for finding a mean.

DISCUSSION

Effective teaching requires not only content knowledge, but also an adequate representation of mathematical ideas and relationships to foster students' understanding. This study found some significant differences as well as interesting similarities in the experienced and inexperienced teachers' conceptions and constructions of pedagogical representations.

AS expected, the first interview task, in which the teachers were asked to predict the solution strategies of 6^{th}- and 7^{th}-grade students, indicated that the experienced teachers' knowledge of both student representations as well as common student errors were more extensive than those of the inexperienced teachers. Whereas the inexperienced teachers tended to think about averaging problems algebraically, the experienced teachers also identified the common student strategies of guess-and-check or visual drawing. The experienced teachers in this study also identified two common student errors, both documented as frequently-committed errors by students in Cai (2000): (1) incorrect use of the averaging algorithm, and (2) unjustified manipulating of numbers. It is interesting to note that although experienced teachers realize possible student errors in solving these problems, they rarely talked about ways to help students overcome them even when they were asked to do so in the interview. The second interview task, in which the teachers were asked to evaluate student solutions, yielded more similar results between the two groups of teachers, although it again revealed a preference for algebraic solutions among the inexperienced teachers, who more often than experienced teachers expressed a belief that the algebraic solutions were more sophisticated than the drawing solutions.

The lesson plan-writing task revealed that although the experienced teachers' knowledge of student representations was more extensive and diverse, this knowledge was not clearly evident in their lesson plans. This puzzling finding could be an artifact of the "outline and worksheet" lesson plan format so commonly used by the experienced teachers. It may be the case, however, that there is some disconnectedness between teachers' knowledge and their planning.

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APPENDIX A

Students' Responses to Two Average-Related Problems

Responses A to C are related to the following Hats Average Problem: Angela is selling hats for the Mathematics Club. (A picture is given showing that 9 hats, 3 hats, and 6 hats were sold in Week, 1, 2, and 3, respectively). How many hats must Angela sell in Week 4 so that the <u>average</u> number of hats sold is 7?

Response A: n = the number of hats sold in Week 4. 9 + 3 + 6 + n = 4 x 7. 18 + n = 28. n = 28 – 18 = 10. Angela sold 10 hats in Week 4.

Response B: The student used "evening-out processes" to solve the problem. The student viewed the average (7) as a leveling basis to "line up" the numbers of hats sold in the Weeks 1, 2, and 3. Since 9 hats were sold in Week 1, that week has two extra hats. Since 3 hats were sold in Week 2, that week needs 4 additional hats in order to line up with the average. Since 6 hats were sold in Week 3, that week needs 1 additional hat to line up with the average. In order to line up with the mean number of hats sold over four weeks, 10 hats should be sold in Week 4.

Response C: 9 + 3 + 6 = 18. 4 x 7 = 28. 28 – 18 = 10. So 10 hats should be sold in Week 4.

Responses D to F are related to the following Score Average Problem: The average of Ed's ten test scores is 87. The teacher throws out the top and bottom scores, which are 55 and 95. What is the average of the remaining set of scores?

Response D: 10 x 87 = 870. 870–55–95 =720. 720/8 = 90. The average of the remaining eight scores is 90.

Response E: The student first used one of the properties of average and determined that the average for the remaining eight scores must be between 55 and 95. Then the student drew ten circles and put 95 in the first and 55 in the last, leaving eight empty circles. Using a modified sharing approach, the student realized that 55 and 95 contributed 15 to the average [(95 + 55) ÷ 10 = 15]. So the student said that each of the eight blank spaces should get 15. But 15 is 72 less than 87 (the average for the ten scores), the student then multiplied 10 by 72 and got 720. 720 ÷ 8 = 90. Thus, 90 became the average of the remaining eight scores after the top and bottom scores were thrown away.

Response F: The student thought that the average for the remaining set of scores must be between 55 and 95. But 87 is closer to 95 than 55. So the average for the remaining eight scores must be about 90.