UNDERSTANDING THE SIGN OF A CORRELATION COEFFICIENT: AN EXAMPLE

Sudhakar Kunte
University of Botswana
Botswana

In many books on Statistics, it is often stated that correlation between two variables X and Y is positive if, as X increases Y also increase. Equivalently, correlation between X and Y is positive, if large values of X most often correspond to the large values of Y and small values X, most often correspond to small values of Y. The correlation is negative if large values of X most often correspond to small values of Y and visa versa. With an example we show that this statement in not always correct. We also give the correct interpretation for the sign of the correlation and its relation to the behavior of the two random variables.

Consider the following experiment: continue tossing a fair coin until, two consecutive heads or two consecutive tails are obtained. Let, X denote the observed total number of Heads and Y denotes the observed total number of Tails in the final observed arrangement. Note that if X = 0, Y has to be two. If X = 1, Y could be 1(the observed sequence HTT) or Y could be 3(the observed sequence THTT). In general, for any value of X > 1, Y can take four possible values as X-2, X-1, X+1 or X+2. The exact value depends upon whether the observed sequence starts with H or T and ends with a HH combination or a TT combination. Thus if X = x we have:

1. Y = x-2, if the observed sequence starts with H and ends with HH.
2. Y = x-1, if the observed sequence starts with T and ends with HH.
3. Y = x+1, if the observed sequence starts with H and ends with TT.
4. Y = x+2, if the observed sequence starts with T and ends with TT.

In any case Y does increase as X increases. Thus we expect the correlation between X and Y to be positive. Kunte and Kharshikar (in press) demonstrate that if the coin which is tossed is a fair coin, then the correlation between X and Y is – 0.2, which is contrary to the intitution. In that paper the authors have also shown that E(X) = E(Y) = 1.5. Thus for the observations on (X , Y), the following pairs have one value above expectation and second value below expectation:
(0 , 2), (2, 0), (1 , 2), (2 ,1), (1 , 3), (3 , 1). Further the combined probability of observing any one of the above pairs is 0.875. Since all these pairs give a negative contribution to the covariance it is not surprising that the final covariance is negative. Thus the correct interpretation for positive correlation is the following:

> *The correlation between X and Y is positive if most often above average values of X correspond to above average values of Y and below average values of X correspond to below average values of Y.*

In this paper we investigate the correlation coefficient between X and Y when instead of tossing a fair coin, we perform a sequence of independent Bernoulli trials with probability of success p as any value in (0 , 1) and we stop the trials as soon two consecutive successes or failures are observed. The joint probability distribution of X and Y is given by:

$P(X = 0, Y = 2) = q^2$

$P(X = 1, Y = 2) = pq^2$

$P(X = 1, Y = 3) = pq^3$

For x > 1,

$P(X = x, Y = x – 2) = p^x q^{x-2}$

$P(X = x, Y = x – 1) = p^x q^{x-1}$

$P(X = x, Y = x +1) = p^x q^{x+1}$

$P(X = x, Y = x + 2) = p^x q^{x-+2}$.

Using these probabilities we can get:

$$E(X) = \frac{p(1+q)}{q}\left\{\frac{(1+q^3)-(1-pq)^2}{(1-pq)^2}\right\},$$

$$E(Y) = \frac{q(1+p)}{p}\left\{\frac{(1+p^3)-(1-pq)^2}{(1-pq)^2}\right\},$$

$$E(XY) = \frac{pq(5-2pq)}{(1-pq)^2}.$$

$$E(X(X-1)) = \frac{2p^2(1+q)(1+q^3)}{(1-pq)^3}.$$

$$E(Y(Y-1)) = \frac{2q^2(1+p)(1+p^3)}{(1-pq)^3}.$$

From these expressions we can obtain the variance of X, variance of Y, covariance between X and Y. Using these expressions we can obtain the correlation coefficient between X and Y. Though the exact formula is difficult to write down, using computer, we have evaluated the correlation coefficient for values of p between .1 to .9. Figure 1 gives the graph of the correlation coefficient as a function of *p*.

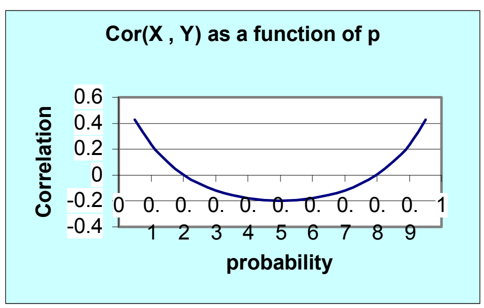

Figure 1. Correlation Coefficient as a Function of *p*

It can be seen that for $.20383 < p < .79617$ the correlation between X and Y is negative and for the extreme values of p the covariance is positive. This change in sign can be explained as follows:

Note that,

$E(X|p=.2) = .514286$ and $E(Y|p=.2) = 2.057143$.

Similarly,

$E(X|p=.3) = .839241$ and $E(Y|p=.3) = 1.958228$.

Now for both $p = .2$ and $p = .3$, the most likely value for (X ,Y) is the pair (0 , 2). For $p = .2$, in the computation of the covariance, the pair (0 , 2) gives a positive contribution, because both X and Y values are below their means. For $p = .3$, the pair (0 , 2) gives the negative contribution, because, in this case X value lies below its mean and Y value lies above its mean. This is the initiative reason as to why there is a change of sign for the covariance between $p = .2$ to $p = .3$.

REFERENCES

Kharshikar A.V., & Kunte S. (in press). Understanding correlation. *Teaching Statistics*.