# Strengths and Limitations of Informal Conceptions in Introductory Probability Courses for Future Lower Secondary Teachers

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## Summary

*In this paper, I describe how and why I changed my approach in teaching probability for future teachers. The main focus is on an introductory activity in probability theory. The way the concept of probability and probability rules are introduced in elementary courses is challenging. I have realized that one has to take into account what students know, or what they seem to know about the subject. Most of the future teachers have intuitive knowledge about probabilities, and we should take their intuitive background seriously. We have to find out in which way this background fits the needs and in which ways it is missed out. The students have adopted these informal conceptions through their use of language while growing up, through conversations and life experiences. By challenging their intuitions by deliberate discussions, objective definitions of probability are constructed jointly in the class. This approach also reveals examples of situations where intuitive knowledge seems to be consistent with objective rules and situations where intuitions and concepts fall apart.*

## Introduction

The following elaborations focus on the design of elementary courses in probability at university college for future teacher students in mathematics. Starting to teach probability in elementary probability courses is unique in the sense that even though future teachers (or students of other disciplines) may never have studied this subject in school, they are in a way prepared to solve simple problems. That is because most people have an intuitive perception of probability. An important task in the teaching is to take advantage of this informal kind of understanding.

Intuitive knowledge in general is defined by Fischbein (1987) as follows:

> “Intuitive knowledge is immediate knowledge; that is, a form of cognition which seems to present itself to a person as being self-evident” (p. 6) .

This means that statements are accepted without feeling the need for a proof. In this paper, I claim that this attitude emerges in persons because they hear and use such concepts in a natural way through developing their language while growing up. Fischbein (1987) states that one of the main characteristics of intuitions is their resistance to change, which could make them an obstacle for establishing objective rules.

> “The perseverance of a certain initial representation may be explained not only by its being the first but also by its fitness to the basic requirements of intuitive cognitions”. (p. 198).

Konold (1995) focuses on the issue that adults have intuitions about probability which are at odds with accepted theories and that these informal conceptions make learning in probability and statistics problematic. He found that some people were outcome-oriented and translated probability values into yes/no decisions. He termed reasoning of this form as the “outcome approach”. Kahnemann and

Tversky (1972) introduced the concept of "judgement heuristics", i. e. rules that people seem to rely on when assessing the likelihood of events. They state:

> "Apparently, people replace the laws of chance by heuristics, which sometimes yield reasonable estimates and quite often do not" (p. 431).

Pfannkuch and Brown (1996) found that with coin-based problems, students explained their answers by invoking probabilistic reasoning. However, the students seemed to be uncomfortable with such thinking, being unable to reconcile this probabilistic approach with their fundamental intuitions. This is in accordance with what Fischbein (1987) claims:

> "We know now that intuitive mechanisms are organized in firm, coherent complex structures very resistant to alterations" (p. 213).

He also speaks about raw primary intuitions which seem to be a way to establish a mathematical model. The raw intuitions are then changed into some sort of secondary intuitions leading to further steps of mathematisation. Borovcnik and Peard (1996) stated that primary intuitions related to probability are often misleading and that teaching without connection to primary intuitions will have no lasting effect. They describe teaching in the form of guided interviews which revealed a lot of causal thinkers. By challenging the causal thinkers' intuitions, they finally succeeded in establishing probabilistic thinking. Except for Fischbein, these authors have a common focus on people's intuitions about probability and subjective probabilities and what problems this may cause in adapting probabilistic thinking.

My research question is to reverse the focus and ask: How and why can we take advantage of people's informal understanding (intuitive knowledge) of probabilities in introductory courses? Before we can give an answer to the first question "how", we need to analyse what this informal knowledge consist of, find its distinctive character and how informal knowledge is developed in probability. By this we may be able to answer the second question "why". That is because there may be a link between this informal understanding, what people seem to know, and the way one may establish objective concepts and rules in probability theory. On the basis of my answers to these questions I made a plan for my first lessons on this topic. This plan was my answer to how we can take advantage of people's informal understanding of probabilities in introductory courses. Furthermore, this plan constitutes the basis of my research.

The way I carried out my research, was to organize the introduction to the topic through problem solving exercises. Both simple and compound events were put forward in these tasks. In this way I could ascertain what the students "knew" about elementary probability models or definitions for simple events. Furthermore, I wanted to see how they managed to find the probabilities of compound events and what arguments they used in their suggestions. The future teachers participating in my investigation had no formal background in probability theory because this subject was not included in the curriculum when they were pupils at school. In this paper I discuss examples of situations where informal understanding seems to lead to right solutions and other examples where informal approaches fail. A study of students' strategies for finding the probabilities of certain compound events reveals a seemingly popular heuristic not yet documented in the literature: Several students find an answer by simply transforming a two-step problem into a one-step problem or simple trial. I have denoted reasoning of this form as "one-step orientated" heuristic.

## Everyday Language, Intuition and Probability

Calculation of probabilities is a topic that differs from most other subjects in mathematics. The most obvious difference is that in probability theory we are dealing with situations of uncertainty in contrast to determinism in pure mathematics. The most important issue in this context, however, is that the concept of probability relates to everyday language, for instance to adjectives like 'probable' and 'likely'. The concept of probability is present in many terms, for instance chance, risk, possible, and sure. In addition, in the language we have adjectives that bring some shading to these expressions, for instance "There is a *small* chance that it will happen".

Such words and expressions are usually not used by children in their first years of schooling. Gradually they will be aware that most of the "things" in this world are neither black nor white. Often they need to express themselves in a careful or modest way, make reservations, and sometimes they are rather insecure about certain issues. Therefore, they need to construct statements that express their uncertainty in their communication with other people. And it is due to this purpose phrases as discussed above are applicable.

As we see, many of these words are related to concepts applied in probability theory and calculations. We have an intuitive interpretation about probability because the word probability is a concept integrated into our use of language. Linguists define these expressions and use of the words probable and probably as *hedges* in the language. In short terms, the use of these hedges helps us to modify the certainty in our statements.

Rowland (2000) describes four categories of hedges to be used in a mathematical context, adapting results from Prince, Frader and Bosk (1982). Words like 'probably', 'think', and 'maybe' are termed to belong to the hedge denoted as Plausibility Shield. The hedges considered in this paper may be attributed to a subdivision of this category. After studying the use of such words or modifiers both in spoken and written language, I find that these hedges may be divided into four different categories.

Category 1. Examples in this category are "The mystery at Boston airport is probably solved." and "He is likely to come." The characteristic of the use of the concept here is that the source of the information is aware that there is a possibility of other outcomes than the one mentioned, but that the one mentioned is expected to be the true outcome.

Category 2. From a linguistic point of view, events in this category are considered to have a smaller or greater chance of occurring. Examples in this category are "It is less likely that she is going to wear those shoes" and "… but it is just now that it seems overwhelmingly probable that this faction is going to join the government after the election". This category is a strengthening of category 1 because of the use of the adjectives, but one is still not sure about the outcome. Nevertheless, I consider that such phrases are used if the source of the information would be highly surprised if the outcome mentioned did not occur.

Category 3. This category is neither a strengthening nor a weakening compared to category 2, but contains a more precise description of what alternatives may occur. The characteristics of this category are that there are at least two outcomes mentioned, and that one of these is said to be more (or less) favourable than others. Examples in this category are "When you hear the news nowadays, you are no longer sure that you will hear a man's voice, on the contrary it is more likely to be a woman's voice" and "A lawyer said that the probability of being shot and killed by a policeman in Dallas is greater than the risk of being murdered by a professional killer in Norway".

Category 4. This category represents a strong accuracy compared to the other categories. The characteristic of this category is the use of numbers to quantify how probable the occurrence of some specified events are. An example in this category is from a weekly magazine stating that "The Smith family had their tenth child, the tenth girl. The chance of something like this happening is estimated to be one in a million". Another example is "The police say that the probability of clearing up this case is 100 percent".

When children are growing up, they hear the use of hedges and expressions like those listed above in social contexts. It is not long before they are able to use them themselves as a natural expansion of their vocabulary. They learn to know that in most areas of life there is more than one possible outcome. Hence, they also get a perception about the concept sample space. Furthermore, through the spoken language they are able to give an argumentation for what outcome seems most likely to happen. When they meet the concept probability at school in an objective form, I assume that most of the children are able to link this new idea to the informal concept they are familiar with. This is an example of the fact that children are stimulated in mathematical concepts outside the school curriculum.

The four categories listed above describe the steps in the linguistic and cognitive development that is needed to make an individual able to interpret expressions like those given above. When they attain this capacity people believe they also have the competence to solve problems belonging to category 4. In other words, these four categories describe the way individuals may develop their intuitive knowledge in probability. "Intuitions are always the product of personal experience, of the personal involvement of the individual in a certain practical or theoretical activity" (Fischbein 1987, p. 213). This is something that the students could take advantage of in their future role of teaching probability.

Going back to the four categories, we see that most people use formulations from all these categories in everyday language, but so do statisticians. In fact, it is difficult to know whether or not the probability mentioned is from a subjective or objective calculation. On the other hand, if one uses just intuition, one will usually be in categories 1-3, whereas the professional statistician is in category 4 most of the time. But the boundaries here are of course not strict.

If people are fully aware of the real meaning of hedges in everyday language, they should be well prepared for lessons in probability also in the sense that every trial has more than one outcome. Even if one event is said to be most likely to happen, one has in mind that the opposite event (complement) also has a certain probability of occurring. Consequently, if one focuses on the meaning in everyday language there should be a good chance to avoid reasoning according to the outcome approach of Konold (1995).

## Teaching Probability Based on Everyday Language

In recent years I follow more or less the same procedure when I start teaching probability to my introductory future teacher classes. After saying hello, I say that now we are about to start with the topic of probability and calculation of probabilities. Furthermore, the students are told that before more is explained about these concepts, they are going to do some exercises on their own. I tell them that these exercises contain well-known problems they should recognise from their daily life. The focus is not on the correct answers to the questions; more important is that the students justify their solutions carefully. Each student may work alone, or discuss with his or her neighbour. If two students collaborate and they find that their solutions are different, they are encouraged to write down both their solutions. I make it clear that I of course do not expect them to solve these exercises formally right at this stage.

What is important for me as a teacher is to find out how they argue for their solutions. I also tell them that after they have finished these exercises, I would read their answers and discuss them in class. In that part, their answers will be used as a basis for discussions in order to establish objective models or concepts of probability. Then they get the exercises and paper to write on. This procedure may seem a bit burdensome for the students, but to my surprise they start working without any questions.

In the beginning, on the very first start of this programme for future teachers, I had the idea of doing something different from giving a traditional and deductive introduction to the topic. Because I was aware of the use of concepts of probability in an informal way in everyday language, I decided to make the introduction through problem solving exercises. That decision resulted in a pilot study, which included exercises 7, 8, and 9 below. When a new group of students started the course the following year, the number of exercises was expanded with six more exercises. Because students do not use the same amount of time on the exercises, four additional exercises were included that would keep the fastest students occupied for the whole period.

Below, I present the first nine and most important out of a total of 13 exercises that were used to answer my research question. At the time of my research, probability was included in the curriculum in school, and exercises 7, 8, and 9 are picked from previous exams in lower secondary school.

1. You are going to toss a fair die; what is the probability of getting a 6?
2. A meteoric stone is falling towards the earth. What would you think is the probability of hitting the USA?

   What do you think is the probability of hitting the ocean?
3. A woman is expecting a baby. What is the probability of having a girl?
4. A family has three children, all girls, and is expecting another baby. The three girls would be pleased if the baby is a boy. What is the probability of having a boy?
5. A deck of cards consists of 52 cards. What is the probability of drawing a spade?

6. You take part in two lotteries A and B. The probability of winning in lottery A is 0.05 (5%), and the probability of winning in lottery B is 0.10 (10%). What is your probability of winning?
7. A family plans to have two children. We assume that the probability of getting a boy is equal to the probability of getting a girl. What is the probability of getting two girls in this family?

8. Two girls and three boys have a dinner party. They agree that two of them are going to do the dishes. They decide it by drawing lots. What is the probability that two boys will do the dishes?
9. Anna has three red, two green and one blue pencil in her pen case. She asks Maria to pick out two pencils without looking. Anna thinks that the probability that both of the pencils are red is 1/5 but Maria thinks that the probability is 1/3. Does either of them have the correct answer?

Note that all problems belong to category 4. In my pilot study, I had a small group of 18; the second study included 58 future teachers. These students were between 20 and 28 years old, with an average age of about 25 years. In their earlier days in school they had neither lectures in probability nor deeper studies in combinations and permutations, except for one student who had a minor course in probability in upper secondary school. When giving exercises like this I find that students collaborate, ask questions, give examples, and discuss their findings and solutions. Thus they also experience the principles of social constructivist learning. After about 45 minutes, the students usually have finished the most important exercises.

In my research study I collected the papers with their answers and explanations. This constitutes my main data which I have analyzed. At that time I also encouraged these students by saying that I am impressed by what they have done so far without any formal teaching in the topic. This test was anonymous. Therefore I neither know gender nor age behind each answer. A discussion of the answers to each exercise and explanations follows.

## Model Building

A characteristic feature of exercises 1-5 is that it is the probability in just one single trial/experiment that is considered. My purpose is to make positive use of the students' intuitive and informal understanding of probability to establish objective models or definitions of probability through discussions. The students may use different ways of thinking or strategies. From the discussions, I found that the lack of relevant explanations in the exercises is not necessarily due to a lack of knowledge.

I succeeded in defining three models or definitions of probability through these discussions with the students.

Everyone found the right answer, 1/6, in exercise 1. In class, I ask "Why is it so"? Typically the student's argument is that the die has 6 sides. For a statistician that's not enough, of course, so I have to stress that these 6 sides are of equal "weight" and that no side is more favourable than the other sides to happen. It helps a lot to get to these conclusions if I show them a matchbox and ask whether they suspect that the probability is 1/6 here, too. I conclude this session by saying that this is a way to state a probability for simple events. We define this as a first model, and call it the geometric or symmetric model (model 1). We assign this name to the model because we stated this probability by exploring the die's geometry.

A large majority of the students use the right principle in exercise 2; they discuss the area considered and find that the probability may be stated as the proportion of the USA (or the ocean, respectively) to the whole surface of the earth. No further arguments for this answer are actually given in the students' answer, except from one student who comes close to the point. In the following discussion I therefore ask questions that lead to the conclusion that we must assume that the power of attraction is equally distributed all over the earth, which physically is a reasonable assumption. In this exercise there are two possibilities, to hit or not hit the USA (the ocean). Note that there is no answer stating that the probability is 50 % which is the answer for people using the primitive strategy if two options are possible.

Exercise 3 happens to be a source of interesting discussions. Most of the students say that we have the probability ½ because there are just two genders and therefore they use model 1 in the same manner as in exercise 1. At this stage it is not clear if some of the students use the primitive strategy mentioned above. Therefore, the students are challenged to be more precise on that issue. There will usually be at least one student that raises a hand and says that in practice one gender is more likely than the other. As a result of the discussion a model of large numbers (model 2) is established, according to which the probability of one event may be seen as the relative frequency in the long run of trials under identical conditions. An important issue is that the students recognize a link between these two models. Given a probability of for instance 0.05 to win in a lottery, means that if you participate under the same conditions every time, you will in the long run win in about 5 % of the trials.

Exercise 4 is not so important at this stage, but I find that almost all the students have the same answer here as in exercise 3. Only three students said that the probability is less than ½. One of those three said the probability is 6.75 %; which is the probability to get four boys in four births. I indicate to the students that the right answer ½ is a result of a concept which is called independence, which means that the probability of having a boy in the fourth birth is not influenced by the actual sex of the first three children born. I am comfortable with the fact that so many students seem to have an informal understanding of this concept.

In exercise 5 the students find another type of challenge. Nevertheless, almost all the students get the right answer. However, the answers are given in different forms; 13/52, ¼ and 25 %. Some of them write 13/52 = 25 % and others ¼ = 25 %. In the discussion that follows they are challenged to tell why they put these numbers in the numerator and the denominator and whether the interpretation of the first fraction is different from the second (the proportion of cards versus the proportion of categories). About 10 % of the students answered 13/52. The majority of the students answer just ¼, they are probably aware of the symmetry in colours. Only three students were wrong in this question.

One of those students answered 1/52. This student may have understood the question as one special spade. When I go through this exercise I also ask under what condition we are allowed to establish these fractions. This question is important because the answer establishes a link to model 1. The conclusion of the discussion about this exercise is that this way of evaluating a probability is an example of what is called the uniform model (model 3).

A characteristic of exercises 1-5 is that the probability in just a single trial/experiment is considered. The purpose is to make a positive use of student's intuitive and informal understanding of probability to establish normative models of probability. The students use many ways of thinking and different strategies. As we have seen, the students are doing well in getting the right answers in these exercises. We have also seen that their explanations are limited. The lack of relevant explanations in the exercises and in the discussions that follow, are not necessarily due to a lack of knowledge. On the contrary, the students may not find it relevant to mention them. In exercise 1 for instance, they may object to this need and say that we all know that all sides of the die are equal. One important task is therefore to encourage students to use precise language in this matter because not all experiments are as obvious as this one.

Now and then some student may object to the separation of model 1 and model 3. They claim that these models are equal. I am pleased when I get such an objection, and I am in a way agreeing with the students. In fact, model 1 is a special case of model 3, but I like to make this distinction at this stage of the teaching process. I find it especially useful when we are dealing with situations where the sample space is not uniform, as in exercise 2.

## Probabilities of Compound Events

As we have seen, the future teachers were also asked to find probabilities of compound events (exercise 6 to 9). On the one hand the situations described in these problems should be familiar to them; on the other hand it is clear that these exercises represent a much bigger technical challenge than the first five. However, the students have no objections to this challenge; they keep on working! The results of the students' calculation showed that the number of correct answers in this part of the test decreased dramatically compared to the number of correct answers in the first part.

At this stage, it is clear that most of the students are unfamiliar with appropriate tools to solve the problems. My conclusion is that the intuitive and informal concepts are sufficient to explore basic probabilities in simple events but are insufficient for calculating probabilities of compound events. I have found several interesting (and erroneous) strategies and explanations students use to find an answer to problems of this kind, and some examples of such strategies will be discussed in the sequel.

In exercise 7 about 40 % give the correct answer of ¼. Some of these students just write down the right answer without any argument. Others show all four combinations that are possible and then write down the right answer, tacitly assuming that each combination has the same probability of occurring. Just 15% believe the right answer is 1/3, the traditionally wrong answer to this problem. The students giving this answer list the three outcomes 2 girls, 1 of each gender and no girls. The answer 1/3 follows because they think the likelihood is the same for all three outcomes. Almost 25% of the students give the answer 50% to this problem. Just one of them gives an explanation for this answer.

- The argument is that since we know there are two girls, the first-born child must be a girl. The second child may then be a girl or a boy. Hence, the probability is 50%. In a way this argument transforms the two-step problem into a one-step problem, but the main reason for this answer is probably that they have not seen that this is a question about the probability of two births.
- Others claim that the probability is 50% because the probability is the same as the probability of getting a boy and a girl. I think this way of reasoning is caused by the same misunderstanding. It might also be the primitive strategy; if there are two possibilities then the answer is 50% for each.
- Some students say this probability is as big as the probability of getting two boys. This argument is correct, but the answer 50% shows that these students neglect, for some reason, the possibility to have one child of each gender.
- Another possibility is that these students also consider this question as a question about the last birth, given a boy in the first birth. By this it is not relevant to consider the possibility to have one child of each gender. These arguments show that the students are rather incomplete in their reasoning.
- The rest, about 20% of the students, avoid answering the question.

The number of correct answers decreases and the number of alternative answers increases in exercise 8. Almost 20% have a right answer of 3/10. Some of these students have first listed the two girls and the three boys on a line. In each line below they successively mark two persons and find that there are 10 distinct possibilities. The answer follows from the fact that two boys (and no girls) are marked in 3 out of these 10 lines.

- Another approach is to connect routes (or lines) between each person in the first line, for instance the first boy may be connected both to the other two boys and to the two girls. Finally they find that there are 10 different routes and that 3 of these routes consist of two boys. The answer 3/10 is obtained by the use of the uniform model in both cases. Here of course it is important to repeat that we are allowed to do so because we assume that each outcome has the same probability to occur.
- More than 15% of the students answer 2/5 or 40%. The argument here is that each person represents 1/5 or 20 %. When the question is about two boys, they simply consider two boys as a unit, each having the probability 1/5 to do the dishes and they add these probabilities. In a way they transform a two-step problem into a single trial and then use the uniform model. This is in accordance with the one-step orientated heuristic.
- Almost 15% of the students answer 3/5, perhaps because of the fact that 3 out of 5 are boys. Also this reasoning neglects the fact that this is a two-step problem. Hence, this is also in accordance with the one-step orientated heuristic. Almost 15% of the students find correctly that the probability of getting a boy in the first draw is 3/5 (60 %) and that the probability to get a boy in the second draw then is 2/4 or 50%. Some leave it that way.
- Others do some calculations and find the answer 55%; the average of the two probabilities. This is due to another primitive strategy; if you have two numbers (probabilities) contained in a task, then add them and divide by 2.
- Consequently some of the students who do not find the right answer, are aware of the fact that this exercise may be considered as a two-step problem, but they fail either in finding

the right number of combinations or in finding out what to do with the two probabilities (60% and 50%).

- The "rest", about 50% of the students, seem to give an incorrect guess or avoid answering the question.

The probabilistic challenge in exercise 9 is almost the same as in exercise 8, but the answers vary in different ways. Here again, about 20% of the students find the right answer of 1/5. Students who have the right answer on exercise 9 do not necessarily have the right answer to exercise 8 and vice versa. This is because they fail to find the right number of combinations in one of these exercises. And this is surprising.

- More than 20% of the students give the answer 1/3 to this exercise. They find this answer by adding 1/6 and 1/6. As we have noted in exercise 8, they have used the one-step orientated heuristic by transforming a two-step problem into a single trial and then use the uniform model.
- Another frequent answer is 50%. This answer may be due to the primitive strategy on the two options "both red" and "not both red". But this answer may also be found by simply considering the proportion of red pencils in the pen case; and by that they also use a one-step orientated heuristic. About 15% of the students give this answer.
- The "rest", about 50 % of the students, seem to give an incorrect guess without further explanations or avoid answering the question.

Finally, I will present the results of exercise 6. The situation described in this exercise should be quite familiar to most people. Nevertheless, this is the exercise that appears to be the most problematic for the students. Less than 4% of the students answer 0.005 which is the right answer for the probability of winning in both lotteries. With the intention of focusing on the need for precise language, this exercise is purposefully made unclear. What is the meaning of "win" here? None of the students have the answer 0.145; this is the probability of winning in at least one of the lotteries.

- The most frequent answer to this problem is 7.5 %; half of the students give this answer. This answer can in no way be right. They find this answer by adding up the two probabilities and dividing the result by two, the same primitive procedure as in exercise 8. No explanation is given for this division, but it may be because they are aware that the probability will be too large by just adding.
- On the other hand, about 15% of the students actually answer 15%, which is higher than to win in each of the two lotteries.
- Another answer is 10%; this is the larger of the two single winning probabilities. 7% of the students give this answer.
- The rest, about 25%, give no answer to this exercise.

This exercise shows that a precise wording of the text is essential. It is reasonable to say that those who have answered 15% have answered the question according to the interpretation of finding the probability of winning in at least one lottery. Those who have answered 7.5 %, however, have perceived the exercise as to find the probability to win in both lotteries.

In this section, I have shown that the future teachers without former formal education in probability theory have much more troubles with problems concerning compound events than with problems concerning simple events. This is not surprising, but it is interesting to watch students trying to solve these problems on the basis of their informal knowledge, which takes the form of a tacit assumption.

Exercises 8 and 9 describe compound events in a different way than exercises 6 and 7. It seems more obvious to the students that the latter exercises are two-step problems than the other two. In exercises 8 and 9 one picks out a concrete sample of two elements from the same concrete population (at the same time!) and this fact seems to mislead many students to consider these problems as one-step problems or simple events. About 1/3 of all the students followed a strategy that might be attributed to this one-step reformulation in exercise 9.

In exercises 7 it seems obvious to the students that the first birth must take place before the second birth, and therefore they consider it as a two-step problem. It is reasonable that they have the same thoughts about the lotteries in exercise 6. Note also that the events considered in exercise 8 and 9 are dependent and that the events considered in exercise 6 and 7 are independent. In exercise 8 and 9 the number of “successes” follows the hypergeometric probability distribution. The number of girls born in exercise 7 is binomially distributed.

Even though the structure of the problems in exercises 6 and 7 are in a way similar, I find that the students’ solutions to these two exercises are quite different. Note for instance that no student tries to find the four combinations in exercise 6 as many do in exercise 7. Neither does any student make a list of three outcomes like 2 win, 1 win and no win here, as 15% of the students do in exercise 7. One hypothesis explaining why we observe this difference in the way students try to solve these problems is that the students in exercise 7 intuitively know that each gender has the same probability of being born, and that the succession of births is more apparent than the ways one may win by participating in two lotteries. Another plausible reason for this difference is that when we talk about births, boys and girls are equally loaded. In lotteries the focus is just on the alternative to win.

This study shows that a common strategy in solving problems with certain compound events seems to be to transform the two-step problem into a simple event and then use the uniform model. This seems to be a common heuristic in problems where one picks out a concrete sample of for instance two elements from the same concrete population (at the same time!), i.e. events that are dependent. If we exclude the students having the right answer and those who have not answered at all, it is clear that this one-step orientated heuristic is the major strategy in finding this probability.

## Implications on Teaching

In recent years our future teachers in introductory courses have more formal background in probability because this subject is now introduced into the curriculum at school. Nevertheless, I have the same introduction as shown in this paper. Most of the exercises serve as a repetition; they also give rich opportunity to discuss potential models as well as the calculations that are done. Many of the problems with compound events are still present. For instance, only a few students are capable of finding adequate answers to exercise 6. The students are keen on getting the right answer on this particular exercise. Sometimes I will not give them the answer immediately because I want to utilize their curiosity as motivation for creating more general concepts, for instance the addition rule involving two independent events.

As described earlier, this way of teaching reveals a lot of solution strategies with which the students are familiar. Of course it is of importance that the teacher is aware of these. I have experienced that a more deductive approach to this subject does not engage students in the way described here, and students' strategies are not revealed to the same degree.

There is no doubt that most of the exercises presented in this article are fit for introductory courses on a lower level, for instance in lower secondary school. Some changes may be necessary, such as replacing the word probability by the word chance. In contrast to a deductive lesson, one may easily achieve a problem-solving lesson followed by valuable discussions.

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## Some Authentic Results of the Students Connected to the One-Step Strategy

Below are selected answers to exercise 8 from future teacher students. A short comment is added in each case.

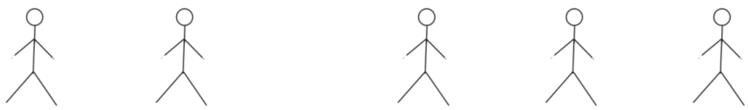
### Exercise 8

> Two girls and three boys have a dinner party. They agree that two of them are going to do the dishes. They decide who by drawing lots. What is the probability that two boys will do the dishes?

**Student 1:** $\frac{1}{5} + \frac{1}{5} = \frac{2}{5}$

*Comment*: Each person has a probability of 1/5 to be drawn in the first draw. Student 1 has probably fixed two of the three boys and added these two probabilities. The complement for instance, is not considered.

**Student 2:**



$\frac{2}{5}$ $\frac{3}{5}$ Probability = $\frac{2}{5}$

*Comment*: Each person has a probability of 1/5 to be drawn in the first draw. Student 2 seems to select two of the three boys and handle them as a unit. The problem is then transformed into a one-step problem. In a way it seems that student 2 has reduced the sample space into two events; either the two boys (on the left) or the three others (on the right, the complement). Student 2 ingnores that the complement hereconsists of three persons, and is fixed on the two boys chosen. Hence, he/she concludes that the probability is 2/5 of drawing these two boys.

**Student 3:**


Each person has a 20% chance to be chosen. Drawing two boys is therefore 2 $\times$20%; i.e. 40%.

*Comment*: Each person has a probability of 20% to be drawn in the first draw. Student 3 seems then to use the same procedure as student 1, except that he/she is multiplying 20% by 2 instead of adding these two numbers.

**Student 4:** There is a chance of 60% to pick out 2 boys to do the dishes because there is 40% chance to pick out a girl.

*Comment*: Student 4 claims that the probability is equal to the proportion of boys in the population, which is the probability to get a boy in the first draw. This is obviously according to the one-step orientated heuristic, but how this student treats the problem of drawing two boys, seems a bit unclear.

**Student 5:** The probability of getting 2 boys is $\frac{3}{5}$ or 60%.

*Comment*: As student 4 student 5 claims that the probability is equal to the proportion of boys in the population. This is obviously according to the one-step orientated heuristic, but how this student treats the problem of drawing two boys, seems also a bit unclear.

**Student 6:** 100:5 = 20. $\frac{2}{5}$ are girls and $\frac{3}{5}$ are boys. $\frac{3}{5} = 0.6 \cdot 100 = 60\%$.

*Comment*: Student 6 has the same answer as student 4 and 5. It seems that all three have the same starting point.

## Some Authentic Results of the Students Connected to the One-Step Strategy

Below are selected answers to exercise 9 from future teacher students. A short comment is added in each case.

### Exercise 9

Anna has three red, two green and one blue pencil in her pen case. She asks Maria to pick out two pencils without looking. Anna thinks that the probability that both of the pencils are red is 1/5 but Maria thinks that the probability is 1/3. Does either of them have the correct answer?

**Student A:** I agree with Maria. Argument: There are a total of 6 pencils and Maria picks out 2 pencils; this leads to the probability $\frac{2}{6} = \frac{1}{3}$ for red pencils.

*Comment*: Each pencil has a probability of 1/6 to be drawn. Student A has probably selected two of the three red pencils, handles these two as a unit and adds these two probabilities. In a way it seems that student A has reduced the sample space into two events; either the two red pencils or the four others. Then student A seems to think there are two possibilities; to pick out those two red pencils or not (the complement). Student A ignores that the complement consists of four pencils.

**Student B:** Maria is right. 2 red pencils out of 6 possible makes $\frac{1}{3}$ ($\frac{2}{6}$).

*Comment*: It seems that student B thinks just like student A.

**Student C:** No, one has to divide the red pencils, 3 pieces, on the number of the total number of pencils, 6 pieces: = $\frac{3}{6}$ = 50 % chance for a red pencil.

*Comment*: The proportion of red pencils is 3/6, and this is the probability for a red pencil in the first draw. Student C leaves it that way, and by that student C has turned the two-step problem into a one-step problem. How this student treats the problem of drawing two boys seems a bit unclear.

**Student D:** If the answer is $\frac{1}{5}$, it ought to be 5 pencils. If the answer is $\frac{1}{3}$, it ought to be just 2 red pencils. It is 3 red of a total of 6. Therefore the answer is 50%.

*Comment*: Student D's reasoning seems not to be consistent. In the first sentence student D seems to fix on one special of the red pencils. In order to get the proportion 1/5, he/she claims that there should be only 5 pencils in all. Student D does not take into account the total number of red pencils. In the next sentences student D seems to have the same strategy as student C. If the proportion is 1/3, just two of the 6 pencils should be red. But since there are 3 red out of 6 pencils, he/she states that the probability is 3/6 = 50 %.

**Student E:** Of the total 100 %, the red pencils are 50 %. Therefore, the probability of getting two red pencils is 50 %.

*Comment*: Student E also thinks that the probability to pick out two red pencils is the proportion of red pencils in the population, which is the probability to pick out a red pencil in the first draw. This strategy is in accordance with the strategy of student C and the last part of student D.