A monotonicity property for a random walk in a partially random environment.

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Abstract

We consider random walks in i.i.d. random environments in $\mathbb{Z}^d$ where only the first d_0 coordinates of the environment are random and the probability of a step in each of the d dimensions is d^{-1}. We show in some particular kinds of random environment that if β_1 is the annealed expected value of the first coordinate of the walk after 1 step, then the annealed speed v_1 in the first coordinate is monotone increasing in β_1, when d is taken sufficiently large compared to d_0

1 Introduction

`sec-intro`

Since the pioneering work of Solomon and others in the mid 1970s to early 1980s, random walk in a random environment (RWRE) has enjoyed somewhat of a revival in recent times as a number of interesting results have been obtained. Many of these results relate to laws of large numbers and invariance principles for i.i.d. random environments that are uniformly elliptic (transition probabilities are bounded away from zero). While the behaviour of one-dimensional RWRE is quite well understood, not much is known about RWRE in high dimensions. A notable exception is the influential paper of Bolthausen, Sznitman and Zeitouni `BSZ03` [1], in which laws of large numbers and central limit theorems are obtained for RWRE in dimensions $d = d_0 + d_1$, where $d_0 \geq 1$ is the number of coordinates in which the environment is random and $d_1 \geq 5$ (so the environment is *partially random*).

In this paper we consider monotonicity properties of the speed for random walks in partially random environments (RWpRE) that are special cases of those considered in `BSZ03` [1]. Such properties have not been extensively studied in the literature, although `Sab04` [5] gives an asymptotic expression for the speed of RWRE as a perturbation of simple random walk. Under certain conditions on the joint distribution of the components of the random environment (essentially, independence or complete dependence), we prove that for RWpRE such that at each site, either the left step or right step is not available, the speed to the right is monotone increasing in the probability that the right step is available. The proof makes use of two main ingredients. The first is the result of `BSZ03` [1] that ensures that the speed exists. The second is the expansion of `HH07` [3], which is valid for all

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annealed RWRE, but is most useful in the case $d_1 \gg d_0$ when one has good control over the terms in the expansion.

Our notation for the general set up for RWRE is as follows:

- $\mathcal{P}$ is the space of probability distributions $(p^{[1]}, \dots, p^{[2d]})$ where each $p_i \geq 0$ and $\sum_{i=1}^{2d} p_i = 1$.
- Given a probability measure μ on $\mathcal{P}$, the random environment is an i.i.d. collection of random probability distributions $\omega = \{\omega_x\}_{x \in \mathbb{Z}^d}$ each with law μ, i.e. $\omega \in \Omega = \mathcal{P}^{\mathbb{Z}^d}$ equipped with the product measure $\mathbb{P} = \mu^{\otimes \mathbb{Z}^d}$.
- The random walk in the random environment ω is the Markov chain $\{S_n\}_{n \geq 0}$ with state space $\mathbb{Z}^d$ and law P_ω such that $P_\omega(S_0 = o) = 1$ and for $i = 1, \dots, d$, $P_\omega(S_{n+1} = S_n + e_i | \vec{S}_n) = \omega_{S_n}^{[2i-1]}$ and $P_\omega(S_{n+1} = S_n - e_i | \vec{S}_n) = \omega_{S_n}^{[2i]}$. Here and throughout, $\vec{S}_n = (S_0, \dots, S_n)$ and $e_1, \dots e_d$ are the standard basis vectors for $\mathbb{R}^d$.
- The so-called annealed law P is given by $P = \mathbb{P} \times P_\omega$, i.e.

$$P(\vec{S}_n = \vec{x}) := \int P_\omega(\vec{S}_n = \vec{x}) d\mathbb{P}(\omega).$$

We restrict ourselves to the situation where $\omega_o^{[i]} = (2d)^{-1}$ for every $i \geq 2d_0$, and $\omega_o^{[2i-1]} + \omega_o^{[2i]} = (2d)^{-1}$ for every $i \leq d_0$, $\mathbb{P}$-almost surely. The latter condition is not essential, but simplifies our analysis. The former is more or less essential as we use simple random walk estimates in the proof of our main result. Unless otherwise specified, we also assume that our RWpRE begins at the origin P-almost surely.

Given a random walk path $\vec{\eta}_m$ (such that $P(\vec{S}_m = \vec{\eta}_m) > 0$) we define a conditional probability measure $P^{\vec{\eta}_m}$ by

e:lawwithhist

$$P^{\vec{\eta}_m}(\vec{S}_n = \vec{y}) := P(\vec{S}_{n+m} = \vec{\eta}_m \circ \vec{y} | \vec{S}_m = \vec{\eta}_m), \tag{1.1}$$

where $\vec{\eta}_m \circ \vec{y}$ denotes the path obtained by adjoining the paths $\vec{\eta}_m$ and $\vec{y}$. Necessarily $y_0 = \eta_m$, so in particular under $P^{\vec{\eta}_m}$ the random walk starts at η_m. Roughly speaking, the results in this paper are obtained by proceeding as if we can ignore the conditioning, and estimating the size of our error in doing so.

In Section sec-mono 2 we discuss the issue of monotonicity for the speed of RWRE as a function of the annealed expected drift at the origin, and state the main result of the paper. In Section sec-transprob 3 we examine the annealed transition probability and its derivative. In Section sec-fromexpansion 4 we review the relevant results from HH07 [3], including the formula for the speed. We bound some of the quantities appearing in this formula, and their derivatives in Section sec-pibd 5 and Section sec-derivpi 6 respectively. Finally in Section sec-proof 7 we prove the main result as a consequence of the given formula for the speed and its derivative.

2 Discussion of monotonicity and the main result

sec-mono

It is fairly natural to wonder if the velocity of a random walk in a partially random environment is monotone increasing in the expected drift at the origin. Experts in RWRE know that this is not the case. In fact it is possible in 1 and high dimensions (probably all dimensions) that the expected drift at the origin and the limiting speed of the RWRE have opposite sign. For example, consider the case $d_0 = d = 1$ where the random environment ω is defined by i.i.d. copies $\{X_x\}_{x\in\mathbb{Z}}$ of a random variable $X \in [0, 1]$, such that $\omega_x^{[1]} = X_x$. If we take $\mathbb{P}(X_o = 1) = .25 = 1-\mathbb{P}(X_o = .25)$ then the (annealed) expected location after the first step is $E[S_1] = .25 + .25 - .75^2 < 0$, but the positive density of sites at which it is impossible to go left ensures that $n^{-1}S_n \to v$ for some $v > 0$, P-almost surely.

Consider a RWpRE where $d_0 = 1$ and $d_1 \geq 0$. Again the random environment ω is defined by i.i.d. copies $\{X_x\}_{x\in\mathbb{Z}}$ of a random variable $X \in [0, 1]$, such that $\omega_x^{[1]} = X_x$. Let $F_X(v) = \mathbb{P}(X \leq v)$. Consider a second i.i.d. random environment defined by $\{\widetilde{X}_x\}_{x\in\mathbb{Z}^d}$, such that $F_{\widetilde{X}}(v) \leq F_X(v)$ for each v. It is easy to couple the random environments and random walks $\{S_n\}_{n\geq 0}$ and $\{\widetilde{S}_n\}_{n\geq 0}$ in those environments such that at each time n, $\widetilde{S}_n^{[i]} = S_n^{[i]}$ for $i > 1$, and $\widetilde{S}_n^{[1]} \geq S_n^{[1]}$. That is, the position of the walks $\{S_n\}_{n\geq 0}$ and $\{\widetilde{S}_n\}_{n\geq 0}$ differ only in the first coordinate and $\widetilde{S}$ is never to the left of S. In particular this implies that if the velocities $v^{[1]} = \lim_{n\to\infty} n^{-1}S_n^{[1]}$ and $\widetilde{v}^{[1]}$ exist (and are non-random) P-almost surely then $\widetilde{v}^{[1]} \geq v^{[1]}$. It is also worth noting that there may be parametric families of environments where monotonicity holds, even though there is no stochastic domination. For example, let $\beta \in [0, 1]$ and $\mathbb{P}(X = \frac{1+\beta}{2}) = \frac{2+\beta}{3} = 1-\mathbb{P}(X = \frac{1-\beta}{2})$. We expect that the speed of a random walk in this environment is monotone increasing in β when the dimension d is sufficiently large.

A natural question to ask is whether this monotonicity in the velocity of the first coordinate remains when $d_0 \geq 2$ and $F_{\widetilde{X}}(v) \leq F_X(v)$ in the first coordinate. In general the answer should be no, as the limiting velocity really does depend on the joint distribution of the coordinates of the random environment. Consider for example the case $d_0 = d = 2$, with X_o and $\widetilde{X}_o$ being independent with identical distribution given by $\mathbb{P}(X_o = 1) = \beta = 1 - \mathbb{P}(X_o = 0)$. The random walk in the two-dimensional i.i.d. random environment given by $(X_o/2, \widetilde{X}_o/2)$ has limiting velocity zero in each coordinate, for all $\beta \in (0, 1)$, since the random walk eventually gets stuck on a finite set of sites. On the other hand one expects that the random walk in the two-dimensional i.i.d. random environment given by $(X_o/2, X_o/2)$ has a non-trivial asymptotic velocity in each coordinate whenever $\beta \neq .5$. Similarly, in this paper we show that when $d_1 \gg d_0 = 2$, the limiting velocity in the first coordinate is continuous and strictly increasing in β whether the distribution of the random environment is given by $(X_o/d, \widetilde{X}_o/d)$ or $(X_o/d, X_o/d)$ above. If one then assumes that for fixed β, the asymptotic velocities are different (say $v_{X,\widetilde{X}} > v_{X,X}$), then the continuity of the velocities as functions of β makes it possible to show that for some $\beta' < \beta$, $v_{X',\widetilde{X}} > v_{X,X}$, where X' is independent of $\widetilde{X}$ and has Bernoulli distribution with parameter β'. In this case the marginal distribution of X stochastically dominates that of X' but the monotonicity property does not hold.

The next natural question then seems to be: In the presence of stochastic domination in the first coordinate, does the monotonicity property (for the velocity of the first coordinate of the walk) hold when the first coordinate of the environment is independent of the rest of the coordinates (holding the joint distribution of the other coordinates of the random environment fixed)? We believe that the answer is yes. We prove this result in high dimensions, in the very special case where the environment in the first coordinate is Bernoulli as above, i.e. from each site in $\mathbb{Z}^d$, either a right step or left step is not available, and the probability that the right step is available at the origin is β. The main result of this paper, which also allows for some coordinates of the environment to be completely dependent on the first, is the following theorem.

`thm:main`

Theorem 2.1. *Let $d = d_1 + d_0$ where $d_0 = d_0^* + d_0' \geq 2$. Let X be a Bernoulli random variable with parameter β. Then there exists $d_c(d_0, d_0^*)$ such that for all $d_1 \geq d_c$ the following holds:*

For any random vector $(X_{d_0^*+1}, \dots, X_{d_0})$ *supported on* $[0,1]^{d_0'}$ *that is independent of* X*, the first coordinate of the speed* $v(\beta)$ *of the random walk in the* d*-dimensional i.i.d. random environment defined by*

e:environ

$$\begin{aligned}\omega_o^{[2i-1]} =& d^{-1}X, \text{ and } \omega_o^{[2i]} = d^{-1}(1-X), \qquad \text{for } i \le d_0^*,\\ \omega_o^{[2i-1]} =& d^{-1}X_i, \text{ and } \omega_o^{[2i]} = d^{-1}(1-X_i), \qquad \text{for } d_0^* < i \le d_0,\\ \omega_o^{[i]} =& (2d)^{-1}, \qquad \text{for } i > 2d_0,\end{aligned} \tag{2.1}$$

is monotone increasing in β*.*

The first condition in (2.1) allows for additional components of the random environment to be completely dependent on the first, while the second allows additional components to be completely independent. A more complicated dependence structure between the first coordinate and other coordinates of the environment leads to ambiguity about what is meant by monotonicity as one changes the distribution of the environment in the first coordinate.

As we have already noted, a much more general monotonicity result than Theorem 2.1 certainly holds in the case $d_0 = 1$ by coupling. A valid criticism of the above theorem is that the class of models to which it applies is rather restricted. In particular the component of the random environment for which monotonicity is being considered is a Bernoulli random variable. However this is a natural setting in which to start investigating monotonicity properties in multidimensional random environments, and it has the advantage that the annealed transition probabilities (given the history) have a particularly nice form whose derivative with respect to β is well behaved.

3 The annealed transition probability.

sec-transprob

In this section we derive an expression for the annealed transition probability

e:anntransprob

$$p^{\vec{\eta}_n}(\eta_n, \eta_{n+1}) := P(S_{n+1} = \eta_{n+1} | \vec{S}_n = \vec{\eta}_n). \tag{3.1}$$

Firstly, observe that by definition,

$$\prod_{i=0}^{n} p^{\vec{\eta}_i}(\eta_i, \eta_{i+1}) = \mathbb{E}\Big[\prod_{i=0}^{n} p_\omega(\eta_i, \eta_{i+1})\Big], \tag{3.2}$$

where $p_\omega(x,y) = \omega_x(y-x)$ is the probability of a transition from x to y in environment ω. This implies that

$$p^{\vec{\eta}_n}(\eta_n, \eta_{n+1}) = \frac{\mathbb{E}\Big[\prod_{i=0}^{n} p_\omega(\eta_i, \eta_{i+1})\Big]}{\mathbb{E}\Big[\prod_{i=0}^{n-1} p_\omega(\eta_i, \eta_{i+1})\Big]}. \tag{3.3}$$

For a directed edge $b = (\underline{b}, \overline{b})$, and a fixed path $\vec{\eta}$, let $\ell_n(b)$ denote the edge local time of the path $\vec{\eta}$ up to time n, i.e.

e:localtime

$$\ell_n(b) = \ell_n(b; \vec{\eta}) = \sum_{i=0}^{n-1} I_{\{(\eta_i, \eta_{i+1}) = b\}}. \tag{3.4}$$

Then

$$\begin{aligned}\mathbb{E}\Big[\prod_{i=0}^{n} p_\omega(\eta_i,\eta_{i+1})\Big] =&\mathbb{E}\Big[\prod_{i=0}^{n}\prod_{b}\Big(1+I_{\{(\eta_i,\eta_{i+1})=b\}}(p_\omega(\eta_i,\eta_{i+1})-1)\Big)\Big]\\ =&\mathbb{E}\Big[\prod_{b}\prod_{i=0}^{n}\Big(1+I_{\{(\eta_i,\eta_{i+1})=b\}}(p_\omega(b)-1)\Big)\Big]=\mathbb{E}\Big[\prod_{b}(p_\omega(b))^{\ell_{n+1}(b)}\Big]\\ =&\mathbb{E}\Big[\prod_{z\in\mathbb{Z}^d}\prod_{b:\underline{b}=z}(p_\omega(b))^{\ell_{n+1}(b)}\Big]=\prod_{z\in\mathbb{Z}^d}\mathbb{E}\Big[\prod_{b:\underline{b}=z}(p_\omega(b))^{\ell_{n+1}(b)}\Big],\end{aligned}\tag{3.5}$$

since we have an i.i.d. environment.

It follows that

$$p^{\vec{\eta}_n}(\eta_n,\eta_{n+1})=\frac{\prod_{z\in\mathbb{Z}^d}\mathbb{E}\Big[\prod_{b:\underline{b}=z}(p_\omega(b))^{\ell_{n+1}(b)}\Big]}{\prod_{z\in\mathbb{Z}^d}\mathbb{E}\Big[\prod_{b:\underline{b}=z}(p_\omega(b))^{\ell_n(b)}\Big]}=\frac{\prod_{z\in\mathbb{Z}^d}\mathbb{E}\Big[\prod_{b:\underline{b}=z}(p_\omega(b))^{\ell_n(b)+I_{\{(\eta_n,\eta_{n+1})=b\}}}\Big]}{\prod_{z\in\mathbb{Z}^d}\mathbb{E}\Big[\prod_{b:\underline{b}=z}(p_\omega(b))^{\ell_n(b)}\Big]}.\tag*{$\overset{\texttt{e:t1}}{(3.6)}$}$$

All terms in the products over z in the numerator and denominator cancel, except for the term where $z=\eta_n$. Thus, ($\overset{\texttt{e:t1}}{3.6}$) reduces to

$$p^{\vec{\eta}_n}(\eta_n,\eta_{n+1})=\frac{\mathbb{E}\Big[\prod_{b:\underline{b}=\eta_n}(p_\omega(b))^{\ell_n(b)+I_{\{(\eta_n,\eta_{n+1})=b\}}}\Big]}{\mathbb{E}\Big[\prod_{b:\underline{b}=\eta_n}(p_\omega(b))^{\ell_n(b)}\Big]}.\tag*{$\overset{\texttt{e:t2}}{(3.7)}$}$$

Equation ($\overset{\texttt{e:t2}}{3.7}$) gives the annealed transition probability for a random walk in an i.i.d. random environment in a general setting. For a unit vector $u\in\mathbb{Z}^d$, and a fixed path $\vec{\eta}$, let $\ell'_n(u)=\ell_n(\eta_n,\eta_n+u)$. For an environment satisfying ($\overset{\texttt{e:environ}}{2.1}$), ($\overset{\texttt{e:t2}}{3.7}$) is equal to

$$\begin{aligned}p^{\vec{\eta}_n}(\eta_n,\eta_{n+1})&=\frac{\mathbb{E}\Big[\prod\limits_{i=1}^{d}(p_\omega(\eta_n,\eta_n+e_i))^{\ell'_n(e_i)+I_{\{\eta_{n+1}=\eta_n+e_i\}}}(\frac{1}{d}-p_\omega(\eta_n,\eta_n-e_i))^{\ell'_n(-e_i)+I_{\{\eta_{n+1}=\eta_n-e_i\}}}\Big]}{\mathbb{E}\Big[\prod\limits_{i=1}^{d}(p_\omega(\eta_n,\eta_n+e_i))^{\ell'_n(e_i)}(\frac{1}{d}-p_\omega(\eta_n,\eta_n-e_i))^{\ell'_n(-e_i)}\Big]}\\ &=\frac{1}{d}\begin{cases}\frac{1}{2}, & \text{if } \eta_{n+1}=\eta_n\pm e_i, i>d_0\\ \dfrac{\mathbb{E}\Big[X^{1+\sum_{i=1}^{d_0^*}\ell'_n(e_i)}(1-X)^{\sum_{i=1}^{d_0^*}\ell'_n(-e_i)}\Big]}{\mathbb{E}\Big[X^{\sum_{i=1}^{d_0^*}\ell'_n(e_i)}(1-X)^{\sum_{i=1}^{d_0^*}\ell'_n(-e_i)}\Big]}, & \text{if } \eta_{n+1}=\eta_n+e_i, i\le d_0^*\\ \dfrac{\mathbb{E}\Big[X^{\sum_{i=1}^{d_0^*}\ell'_n(e_i)}(1-X)^{1+\sum_{i=1}^{d_0^*}\ell'_n(-e_i)}\Big]}{\mathbb{E}\Big[X^{\sum_{i=1}^{d_0^*}\ell'_n(e_i)}(1-X)^{\sum_{i=1}^{d_0^*}\ell'_n(-e_i)}\Big]}, & \text{if } \eta_{n+1}=\eta_n-e_i, i\le d_0^*\\ \dfrac{\mathbb{E}\Big[\prod\limits_{i=d_0^*+1}^{d_0}X_i^{\ell'_n(e_i)+I_{\{\eta_{n+1}=\eta_n+e_i\}}}(1-X_i)^{\ell'_n(-e_i)+I_{\{\eta_{n+1}=\eta_n-e_i\}}}\Big]}{\mathbb{E}\Big[\prod\limits_{i=d_0^*+1}^{d_0}X_i^{\ell_n(e_i)}(1-X_i)^{\ell_n(-e_i)}\Big]}, & \text{if } \eta_{n+1}=\eta_n\pm e_i, d_0^*<i\le d_0.\end{cases}\end{aligned}\tag*{$\overset{\texttt{e:cases}}{(3.8)}$}$$

The first and last cases in $\overset{\texttt{e:cases}}{(3.8)}$ do not depend on β, while the first does not even depend on $\vec{\eta}_n$. In each of the middle two expressions, at least one of the sums in the exponents has to be zero, since at each location either all of the steps $+e_i$, $i \leq d_0^*$ are unavailable, or all of the steps $-e_i$, $i \leq d_0^*$ are unavailable. Let $L_n^+ = \sum_{i=1}^{d_0^*} \ell'_n(e_i)$ and $L_n^- = \sum_{i=1}^{d_0^*} \ell'_n(-e_i)$. Then at least one of L_n^+ or L_n^- is zero, and the middle terms of $\overset{\texttt{e:cases}}{(3.8)}$ are

$$\begin{aligned} p^{\vec{\eta}_n}(\eta_n, \eta_n + e_i) =& d^{-1} \frac{\mathbb{E}\Big[X^{L_n^+ + 1}(1-X)^{L_n^-}\Big]}{\mathbb{E}\Big[X^{L_n^+}(1-X)^{L_n^-}\Big]} = d^{-1}\Big(I_{\{L_n^+ > 0, L_n^- = 0\}} + \beta I_{\{L_n^+ = 0, L_n^- = 0\}}\Big) \\ p^{\vec{\eta}_n}(\eta_n, \eta_n - e_i) =& d^{-1} - p^{\vec{\eta}_n}(\eta_n, \eta_n + e_1). \end{aligned}$$

It follows immediately that the derivative $\frac{\partial}{\partial \beta} p^{\vec{\eta}_n}(\eta_n, \eta_n + u)$ exists and satisfies

$$\left| \frac{\partial}{\partial \beta} p^{\vec{\eta}_n}(\eta_n, \eta_n + u) \right| \leq \frac{1}{d} I_{\{L_n^+ = 0 = L_n^-\}} I_{\{u = \pm e_i, i \leq d_0^*\}} \leq \frac{1}{d} I_{\{u = \pm e_i, i \leq d_0^*\}}. \tag*{$\overset{\texttt{e:delp}}{(3.9)}$}$$

4 Results of the lace expansion

`sec-fromexpansion`

In the present setting of RWpRE, the velocity $v := \lim_{n\to\infty} n^{-1} S_n$ has been proved $\overset{\texttt{BSZ03}}{[1]}$ to exist P-almost surely. By Proposition 3.1 of $\overset{\texttt{HH07}}{[3]}$, the speed is given by

$$v = E[S_1] + \sum_{m=2}^{\infty} \sum_x x \pi_m(x), \tag*{$\overset{\texttt{e:theta}}{(4.1)}$}$$

provided this series converges. Here $\pi_m(\cdot)$ is a function (defined below) that depends on β, d_0, d_0^*, d and on the particular distribution of the X_i, for $i = d_0^* + 1, \dots, d_0$.

The function π_m involves the following factors. For $i \geq 1$, let

$$\Delta^{(i)}_{j_i+1} = \Big(p^{\vec{\eta}^{(i-1)}_{j_{i-1}+1} \circ \vec{\eta}^{(i)}_{j_i}} - p^{\vec{\eta}^{(i)}_{j_i}}\Big)(\eta^{(i)}_{j_i}, \eta^{(i)}_{j_i+1}), \tag*{$\overset{\texttt{e:Deltaidef}}{(4.2)}$}$$

with $j_0 \equiv 0$.

Define $\mathcal{A}_{m,N} = \{(j_1, \dots, j_N) \in \mathbb{Z}_+^N : \sum_{l=1}^N j_l = m - N - 1\}$, $\mathcal{A}_N = \dot{\bigcup}_m \mathcal{A}_{m,N}$ and

$$\begin{aligned} &\pi_m^{(N)}(x,y) \\ =& \sum_{\vec{j} \in \mathcal{A}_{m,N}} E^{\varnothing}\Bigg[E^{\vec{\eta}_1^{(0)}}\Bigg[\sum_{\eta^{(1)}_{j_1+1}} \Delta^{(1)}_{j_1+1} \, E^{\vec{\eta}^{(1)}_{j_1+1}}\Bigg[\sum_{\eta^{(2)}_{j_2+1}} \Delta^{(2)}_{j_2+1} \cdots E^{\vec{\eta}^{(N-1)}_{j_{N-1}+1}}\Bigg[\sum_{\eta^{(N)}_{j_N+1}} \Delta^{(N)}_{j_N+1} I_{\{\eta^{(N)}_{j_N} = x, \eta^{(N)}_{j_N+1} = y\}}\Bigg] \cdots \Bigg]\Bigg]\Bigg], \end{aligned} \tag*{$\overset{\texttt{e:piNxydef}}{(4.3)}$}$$

where $E^{\vec{\eta}_m}$ denotes expectation with respect to a conditional probability measure $P^{\vec{\eta}_m}$ defined in $\overset{\texttt{e:lawwithhist}}{(1.1)}$.

We then define

$$\pi_m(x,y) = \sum_{N=1}^{\infty} \pi_m^{(N)}(x,y), \quad \pi^{(N)}(x,y) = \sum_m \pi_m^{(N)}(x,y), \quad \text{and} \quad \pi_m(y) = \sum_{N=1}^{\infty} \sum_x \pi_m^{(N)}(x,y). \tag*{$\overset{\texttt{e:piotherdef}}{(4.4)}$}$$

A difference in transition probabilities (see $\overset{\texttt{e:Deltaidef}}{(4.2)}$) with two different histories $\vec{\eta}^{(i-1)}_{j_{i-1}} \circ \vec{\eta}^{(i)}_{j_i}$ and $\vec{\eta}^{(i-1)}_{j_{i-1}+1}$ occurs precisely when one of the following holds:

- the proposed step $\eta^{(i)}_{j_i+1} - \eta^{(i)}_{j_i}$ is equal to $\pm e_m$ for some $m \in \{d_0^*+1, \dots, d_0\}$, and one of the edges $(\eta^{(i)}_{j_i}, \eta^{(i)}_{j_i} \pm e_m)$ for $m \in \{d_0^*+1, \dots, d_0\}$, was traversed by $\vec{\eta}^{(i-1)}_{j_{i-1}+1}$, or
- the proposed step $\eta^{(i)}_{j_i+1} - \eta^{(i)}_{j_i}$ is equal to $\pm e_m$ for some $m \le d_0^*$, and one of the edges $(\eta^{(i)}_{j_i}, \eta^{(i)}_{j_i} \pm e_m)$ for $m \le d_0^*$ was traversed by $\vec{\eta}^{(i-1)}_{j_{i-1}+1}$.

From `e:cases` (3.8)

`e:Deltabound`

$$|\Delta^{(i)}_{j_i+1}| \le \frac{1}{d} I_{\{(\eta^{(i)}_{j_i}, \eta^{(i)}_{j_i+1}) \in \vec{\eta}^{(i-1)}_{j_{i-1}+1}\}} I_{\{\eta^{(i)}_{j_i+1} = \eta^{(i)}_{j_i} = \pm e_m, m \le d_0\}}. \tag{4.5}$$

Moreover, as in the discussion after `e:cases` (3.8), the derivative $\frac{\partial}{\partial\beta}\Delta^{(i)}_{j_i+1}$ exists and satisfies

`e:delDelta`

$$\begin{aligned} \left|\frac{\partial}{\partial\beta}\Delta^{(i)}_{j_i+1}\right| \le & d^{-1} I_{\{\eta^{(i)}_{j_i+1} - \eta^{(i)}_{j_i} = \pm e_m, m \le d_0^*\}} \left[I_{\{L^+_{j_{i-1}+1+j_i} > 0, L^-_{j_{i-1}+1+j_i} = 0\}} + I_{\{L^+_{j_{i-1}+1+j_i} = 0, L^-_{j_{i-1}+1+j_i} > 0\}} \right] \\ \le & d^{-1} I_{\{\eta^{(i)}_{j_i+1} - \eta^{(i)}_{j_i} = \pm e_m, m \le d_0^*\}} I_{\{(\eta^{(i)}_{j_i}, \eta^{(i)}_{j_i+1}) \in \vec{\eta}^{(i-1)}_{j_{i-1}+1}\}}, \end{aligned} \tag{4.6}$$

where $L^+_{j_{i-1}+1+j_i}$ and $L^-_{j_{i-1}+1+j_i}$ are the cumulative edge local times (defined below `e:cases` (3.8)) for the concatenated walk $\vec{\eta}^{(i-1)}_{j_{i-1}+1} \circ \vec{\eta}^{(i)}_{j_i}$.

Strategy of the proof of Theorem `thm:main` 2.1. We follow the analysis in `HH08mono` [4]. We fix d, d_0, d_0^* and the distribution of $(X_{d_0^*+1}, \dots, X_{d_0})$, differentiate the right hand side of `e:theta` (4.1) with respect to β, and prove that this derivative is positive for all $\beta \in [0,1]$, when d is sufficiently large. From `e:theta` (4.1) and using the fact that $\sum_y \pi_m(x,y) = 0$ (recall `e:piotherdef` (4.4)), we have

`e:theta2`

$$\theta_1 = \frac{2\beta - 1}{d} + \sum_{m=2}^{\infty} \sum_{N=1}^{\infty} \sum_{x,y} (y_1 - x_1) \pi_m^{(N)}(x,y). \tag{4.7}$$

Letting $\varphi_m^{(N)}(x,y) = \frac{\partial}{\partial\beta}\pi_m^{(N)}(x,y)$ and assuming that the limit can be taken through the infinite sums, we then have

`e:deriv1`

$$\frac{\partial\theta_1}{\partial\beta} = \frac{2}{d} + \sum_{N=1}^{\infty} \sum_{m=2}^{\infty} \sum_{x,y} (y_1 - x_1) \varphi_m^{(N)}(x,y). \tag{4.8}$$

Since $\varphi_m^{(N)}(x,y) \equiv 0$ unless $|x-y| = 1$, we have that

`e:needed1`

$$\left| \frac{\partial\theta_1}{\partial\beta} - \frac{2}{d} \right| \le \sum_{N=1}^{\infty} \sum_{m=2}^{\infty} \sum_{x,y} |\varphi_m^{(N)}(x,y)|. \tag{4.9}$$

We conclude that $\frac{\partial\theta_1}{\partial\beta}$, is positive for any β at which $\sum_{N=1}^{\infty}\sum_{m=2}^{\infty}\sum_{x,y} |\varphi_m^{(N)}(x,y)| < 2d^{-1}$. This is what we shall prove in the remainder of this paper, which is organised as follows. In Section `sec-pibd` 5, we start by proving bounds on $\pi_m^{(N)}$. These bounds will be crucially used to prove bounds on $\varphi_m^{(N)}$ in Section `sec-derivpi` 6.

5 Bound on π

sec-pibd

thm:main HHO8mono lem:decomposition lem:tograeens

Before proceeding to the proof of Theorem 2.1, we prove a bound on $\sum_{x,y}\sum_m |\pi^{(N)}_m(x,y)|$ for any self-interacting random walk that satifies certain properties. The proof is an adaption of that in [4] in the context of excited random walk, and makes use of Lemmas 5.1 and 5.2 below. For the first of these lemmas we need to introduce some notation.

Let $f_{i,j_i}(\vec{\eta}^{(i-1)}_m,\vec{\eta}^{(i)}_{j_i}) \geq 0$, $i=0,\dots,N$, be measurable functions from the set of (ordered) pairs of finite random walk paths $(\vec{\eta}^{(i-1)}_m,\vec{\eta}^{(i)}_{j_i})$ such that $m<\infty$ and $\eta^{(i-1)}_m=\eta^{(i)}_0$ (the former is defined to be the origin if $\vec{\eta}^{(i-1)}_m=\varnothing$). Recall that $E^{\vec{\eta}^{(i-1)}_m}$ denotes expectation with respect to the annealed law of RWpRE with given (finite) history $\vec{\eta}^{(i-1)}_m$ (i.e., conditional on the first m steps of the walk being $\vec{\eta}^{(i-1)}_m$). We write $E^{l,\vec{\eta}^{(i-1)}_m}$ to distinguish expectation with respect to different laws (indexed by l), i.e., if $l\neq r$ then $P^{l,\vec{\eta}^{(i-1)}_m}$ and $P^{r,\vec{\eta}^{(i-1)}_m}$ may be different self-interacting random walk laws (with the same given history).

Given $\vec{f}_N=(f_{0,j_0},f_{1,j_1},\dots,f_{N,j_N})$, $k\in\{0,\dots,N\}$, and laws $P^{l,\vec{\eta}^{(i-1)}_m}$, $l\in\mathbb{N}$, we define

e:bigaltpidef

$$\begin{aligned}\Pi^{(k)}_N(\vec{f}_N) := \sum_{\vec{j}\in\mathcal{A}_N} E^{\varnothing}\Bigg[& f_{0,j_0}(\varnothing,\vec{\eta}^{(0)}_{j_0})E^{\vec{\eta}^{(0)}_{j_0+1}}\Big[f_{1,j_1}(\vec{\eta}^{(0)}_{j_0+1},\vec{\eta}^{(1)}_{j_1})\cdots\sum_{l=1}^{j_k}E^{l,\vec{\eta}^{(k-1)}_{j_{k-1}+1}}[f_{k,j_k}(\vec{\eta}^{(k-1)}_{j_{k-1}+1},\vec{\eta}^{(k)}_{j_k})\dots \\ & E^{\vec{\eta}^{(N-1)}_{j_{N-1}+1}}[f_{N,j_N}(\vec{\eta}^{(N-1)}_{j_{N-1}+1},\vec{\eta}^{(N)}_{j_N})]\dots\Big]\Bigg].\end{aligned} \tag{5.1}$$

HHO8mono

We further let $\Pi_N(\vec{f}_N)$ be identical to $\Pi^{(k)}_N(\vec{f}_N)$, apart from the fact that $\sum_{l=1}^{j_k}E^{l,\vec{\eta}^{(k-1)}_{j_{k-1}+1}}$ is replaced with $E^{\vec{\eta}^{(k-1)}_{j_{k-1}+1}}$. A crucial ingredient in obtaining bounds on lace expansion coefficients is the following result, whose elementary proof appears in [4].

lem:decomposition HHO8mono

Lemma 5.1 (Lemma 3.1 of [4]). *Let $\vec{\eta}^{(0)},\dots,\vec{\eta}^{(N)}$ be any collection of N self-interacting random walks defined on the same probability space $(\Omega,\mathcal{F},P)$. Suppose that $f_{i,j_i}\geq 0$, $i=0,\dots,N$ are such that for each $i=0,\dots,N$ there exist constants $K_i\geq 0$, and functions $\kappa_i\geq 0$ (with $\kappa_{-1}\equiv 1$) such that*

e:piecebound

$$\sum_{j_i=0}^{\infty}\kappa_i(j_i)E^{\vec{\eta}^{(i-1)}_m}[f_{i,j_i}(\vec{\eta}^{(i-1)}_m,\vec{\eta}^{(i)}_{j_i})]\leq K_i\kappa_{i-1}(m), \tag{5.2}$$

for each m, uniformly in $\vec{\eta}^{(i-1)}_m$. Then

e:PiNbd

$$\Pi_N(\vec{f}_N)\leq\prod_{i=0}^{N}K_i. \tag{5.3}$$

e:PiNbd e:piecebound

The conclusion in (5.3) also holds for $\Pi^{(k)}_N(\vec{f}_N)$ if there exist $K_i,\kappa_i\geq 0$ such that (5.2) holds for $i\neq k$, and for $i=k$,

e:altpiecebound

$$\sum_{j_k=0}^{\infty}\kappa_k(j_k)\sum_{l=1}^{j_k}E^{l,\vec{\eta}^{(k-1)}_m}[f_{k,j_k}(\vec{\eta}^{(k-1)}_m,\vec{\eta}^{(k)}_{j_k})]\leq K_i\kappa_{k-1}(m), \tag{5.4}$$

for each m, uniformly in $\vec{\eta}^{(i-1)}_m$.

Let $D_d(x) = I[|x| = 1]/(2d)$ denote the simple random walk step distribution. Let $f^{*k}(x)$ denote the k-fold convolution of f with itself, where the convolution of (absolutely summable) functions f, g on $\mathbb{Z}^d$ is defined by

$$(f * g)(x) = \sum_y f(y)g(x - y). \tag{5.5}$$

Then let $G_d(x) = \sum_{k=0}^{\infty} D_d^{*k}(x)$ denote the Green's function for this random walk.

Recall that $d = d_1 + d_0$ where the environment is random only in the d_0 coordinates, and that $P(\omega_{n+1} \in \{\omega_n \pm e_i\}|\vec{\omega}_n) = d^{-1}$ for each $i \geq 0$. The following Lemma, in which

e:Eidef

$$\mathcal{E}_i(d_1, q) = \sup_{v \in \mathbb{Z}^{d_1}} \Big(q^{-(i+1)} G_{d_1}^{*(i+1)}(v) - \delta_{0,v} \Big), \tag{5.6}$$

for $i \geq 0$, immediately applies to our RWpRE with $q = \frac{d_1}{d}$.

lem:togreens

Lemma 5.2 (Diagrammatic bounds). *For any self-interacting random walk $\vec{\omega}$ in $d = d_1 + d_0$ dimensions with the properties that*

- *there exists $q > 0$ such that $P^{\vec{\eta}_m}\Big(S_{n+1} \in \{S_n \pm e_i, i > d_0\}\Big) \geq q$ for all n and $\vec{\eta}_m$, and*
- *the sequence of steps in the d_1 coordinate directions constitute simple random walk steps in d_1 dimensions,*

we have

e:Gbound1

$$\sum_{j=0}^{\infty} \frac{(j+i)!}{j!} P^{\vec{\eta}_m}(S_j = u) \leq i! q^{-(i+1)} G_{d_1}^{*(i+1)}(0), \tag{5.7}$$

e:Gbound2

$$\sum_{j=1}^{\infty} \frac{(j+i)!}{j!} P^{\vec{\eta}_m}(S_j = u) \leq i! \mathcal{E}_i(d_1, q). \tag{5.8}$$

for $i \geq 0$, uniformly in $u \in \mathbb{Z}^d$.

Proof. Exactly as in the proof of HH08mono [4]Lemma 3.2, replacing $\frac{d}{d-1}$ by q^{-1} and $d - 1$ by d_1.

□

Define

e:addef

$$a_{d,d_1} := \frac{d}{d_1^2} G_{d_1}^{*2}(0), . \tag{5.9}$$

prp:pibound

Proposition 5.3 (Two bounds on the expansion coefficients). *For RWpRE, we have*

e:pibound

$$\sum_{x,y} \sum_m |\pi_m^{(N)}(x, y)| \leq \begin{cases} \frac{d_0 \mathcal{E}_0(d_1, d_1 d^{-1})}{d^2}, & N = 1, \\ \frac{d_0 G_{d_1}(0)}{d^2 d_1} \mathcal{E}_1(d_1, d_1 d^{-1}) \left(a_{d,d_1}\right)^{(N-2)}, & N \geq 2. \end{cases} \tag{5.10}$$

It then follows from Theorem 1.4 of BSZ03 [1], Proposition 3.1 of HH07 [3] and the fact that $G_5(0) < \infty$, that e:theta (4.1) holds provided $d_1 \geq 5$ and $a_{d,d_1} < 1$.

Let

e:fdefs

$$\begin{aligned} f_{0,j_0}(\vec{\eta}_m,\vec{S}_{j_0}) =& I_{\{j_0=0\}} I_{\{S_{j_0+1}\in\{S_{j_0}\pm e_r, r\le d_0\}\}}, \\ f_{1,j_1}(\vec{\eta}_m,\vec{S}_{j_1}) =& \frac{1}{d} I_{\{j_1 \text{ is odd}\}} \sum_{S_{j_1+1}} I_{\{(S_{j_1},S_{j_1+1})=(\eta_0,\eta_1)\}}, \\ f_{i,j_i}(\vec{\eta}_m,\vec{S}_{j_i}) =& \frac{1}{d} \sum_{S_{j_i+1}} I_{\{(S_{j_i},S_{j_{i+1}})\in\{(\eta_l,\eta_{l+1}):0\le l<m\}\}}, \quad \text{for } i>1, \quad \text{and} \end{aligned} \tag{5.11}$$

We use Lemma 5.1, together with the following lemma to prove Proposition 5.3.

lem:fbounds

Lemma 5.4. *For RWpRE, with* f_{i,j_i} *defined in* (5.11)

e:fbounds

$$\begin{aligned} &\sum_{j_0=0}^{\infty}(j_0+1)E^{\vec{\eta}_m}[f_{0,j_0}] \le \frac{d_0}{d}, \qquad \sum_{j_1=0}^{\infty}(j_1+1)E^{\vec{\eta}_m}[f_{1,j_1}] \le \frac{1}{d}\mathcal{E}_1(d_1,d_1d^{-1}), \\ &\sum_{j_1=0}^{\infty} E^{\vec{\eta}_m}[f_{1,j_1}] \le \frac{1}{d}\mathcal{E}_0(d_1,d_1d^{-1}), \qquad \sum_{j_i=0}^{\infty}(j_i+1)E^{\vec{\eta}_m}[f_{i,j_i}] \le m a_{d,d_1}, \quad i=2,\dots,N-1, \\ &\sum_{j_N=0}^{\infty} E^{\vec{\eta}_m}[f_{N,j_N}] \le m\frac{1}{d_1}G_{d_1}(0). \end{aligned} \tag{5.12}$$

Proof. The first bound is trivial. For the second bound, since $\sum_{S_{j+1}} I_{\{S_{j+1}=\eta_{l+1}\}} = 1$ and the conditions that $j \ge 0$ and j is odd imply that $j \ge 1$, we have

$$\begin{aligned} \sum_{j=0}^{\infty}(j+1)E_1^{\vec{\eta}_m}[f_{1,j}] \le& \frac{1}{d}\sum_{j=1}^{\infty}(j+1)E^{\vec{\eta}_m}[I_{\{S_j=\eta_0\}}] \\ =& \frac{1}{d}\sum_{j=1}^{\infty}(j+1)P^{\vec{\eta}_m}(S_j=\eta_0) \le \frac{1}{d}\mathcal{E}_1(d_1,d_1d^{-1}), \end{aligned} \tag{5.13}$$

where the last inequality holds by (5.8) with $i=1$. Similarly (5.8) with $i=0$ gives us the third bound.

For the fourth bound, using

$$I_{\{(S_{j_i},S_{j_{i+1}})\in\{(\eta_l,\eta_{l+1}):0\le l<m\}\}} \le \sum_{l=0}^{m-1} I_{\{S_{j_i}=\eta_l\}} I_{\{S_{j_i+1}=\eta_{l+1}\}}$$

and proceeding as for the second bound we see that

$$\sum_{j_i=0}^{\infty}(j_i+1)E^{\vec{\eta}_m}[f_{i,j_i}] \le \sum_{l=0}^{m-1}\frac{1}{d}\sum_{j=0}^{\infty}(j+1)P^{\vec{\eta}_m}(S_j=\eta_l) \le m a_{d,d_1},$$

where we have used (5.7) with $i=1$ in the last step.

For the fifth, proceed as above to get

$$\sum_{j=0}^{\infty} E_N^{\vec{\eta}_m}[f_{N,j}] \le \frac{1}{d} m \sup_u \sum_{j_N=0}^{\infty} P^{\vec{\eta}_m}(S_{j_N}=u) \le m\frac{1}{d_1}G_{d_1}(0),$$

where the last inequality holds by (5.7) with $i = 0$.

□

Proof of Proposition 5.3. It follows from (4.3) and (4.5) that for all $N \geq 1$,

$$\sum_{x,y}\sum_{m} |\pi_m^{(N)}(x,y)| \leq \Pi_N(\vec{f}_N). \tag{5.14}$$

where the functions f_{i,j_i} in $\vec{f}_N$ are given by (5.11).

If $N = 1$ then applying Lemma 5.1 to (5.14) with $\kappa_1 = 1$, $K_0 = \frac{1}{d}$ and $K_1 = \frac{1}{d}\mathcal{E}_0(d_1, d_1 d^{-1})$ (i.e. the right hand side of the third bound of (5.12)) we easily obtain the result.

For $N \geq 2$, applying Lemma 5.1 to (5.14) with $\kappa_N = 1$, $\kappa_i(j_i) = (j_i + 1)$ for $i \neq N$, and

$$K_0 = \frac{1}{d}, \quad K_1 = \frac{1}{d}\mathcal{E}_1(d_1, d_1 d^{-1}), \quad K_N = \frac{1}{d_1}G_{d_1}(0), \quad \text{and} \quad K_i = a_{d,d_1}, \text{ for } 2 \leq i \leq N-1 \tag{5.15}$$

(see the right hand sides of the remaining bounds of (5.12)), we obtain the result. □

6 The differentiation step

From (4.3) we have

$$\pi_m^{(N)}(x,y) = \sum_{\vec{j}\in\mathcal{A}_{m,N}} \sum_{\vec{\eta}_1^{(0)}} \sum_{\vec{\eta}_{j_1+1}^{(1)}} \cdots \sum_{\vec{\eta}_{j_N+1}^{(N)}} I_{\{\eta_{j_N}^{(N)}=x,\eta_{j_N+1}^{(N)}=y\}} p^{\varnothing}(0,\eta_1^{(0)}) \prod_{n=1}^{N}\prod_{i_n=0}^{j_n-1} p^{\vec{\eta}_{j_{n-1}+1}^{(n-1)}\circ\vec{\eta}_{i_n}^{(n)}}\left(\eta_{i_n}^{(n)}, \eta_{i_n+1}^{(n)}\right)\Delta_{j_n+1}^{(n)}. \tag{6.1}$$

Recall that

$$\varphi_m^{(N)}(x,y) := \frac{\partial}{\partial\beta}\pi_m^{(N)}(x,y). \tag{6.2}$$

To verify the exchange of limits involved in taking the derivative with respect to β_1 inside the infinite series defining the formula for the speed (4.1), it is sufficient to prove that $\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)$ is absolutely summable in m and N and that $\sum_{N=1}^{\infty}\sum_{m=2}^{\infty}\sup_{\beta\in[0,1]}|\sum_{x,y}(y-x)\varphi_m^{(N)}(x,y)| < \infty$. By Proposition 5.3 and the fact that $|y-x| = 1$ for x, y nearest neighbours, the first condition holds provided that

$$\boxed{a_{d,d_1} < 1}. \tag{6.3}$$

In fact we will see later on that this inequality is sufficient to also establish the second condition. We now identify $\varphi_m^{(N)}(x,y)$.

We write,

$$\varphi_m^{(N)}(x,y) = \varphi_m^{(N,1)}(x,y) + \varphi_m^{(N,2)}(x,y) + \varphi_m^{(N,3)}(x,y), \tag{6.4}$$

where (by Leibniz' rule), $\varphi_m^{(N,1)}(x,y)$, $\varphi_m^{(N,2)}(x,y)$ and $\varphi_m^{(N,3)}(x,y)$ arise from differentiating $p^{\varnothing}(0,\eta_1^{(0)})$, $\prod_{n=1}^{N}\prod_{i_n=0}^{j_n-1} p^{\vec{\eta}_{j_{n-1}+1}^{(n-1)}\circ\vec{\eta}_{i_n}^{(n)}}\left(\eta_{i_n}^{(n)}, \eta_{i_n+1}^{(n)}\right)$ and $\prod_{n=1}^{N}\prod_{i_n=0}^{j_n-1}\Delta_{j_n+1}^{(n)}$, respectively.

Let $\rho^{(N)}$ be obtained by replacing $p^{\varnothing}(0,\eta_1^{(0)})$ in (6.1) with $d^{-1}I_{\{\eta_1^{(0)}=\pm e_r, r\le d_0^*\}}$ (a bound on its derivative) and by bounding $\Delta^{(n)}_{j_n+1}$ by $|\Delta^{(n)}_{j_n+1}|$ for all $n=1,\dots,N$.

For $k=1,\dots,N$, let $\gamma_k^{(N)}$ be obtained from (6.1) by bounding $\Delta^{(n)}_{j_n+1}$ by $|\Delta^{(n)}_{j_n+1}|$ for all $n=1,\dots,N$ and by replacing $\prod_{i_k=0}^{j_k-1} p^{\vec{\eta}^{(n-1)}_{j_{n-1}+1}\circ\vec{\eta}^{(k)}_{i_k}}\left(\eta^{(k)}_{i_k},\eta^{(k)}_{i_k+1}\right)$ with the following bound on its derivative

$$\sum_{l=0}^{j_k-1}\frac{I_{\{\eta^{(k)}_{l+1}-\eta^{(k)}_{l}=\pm e_r, r\le d_0^*\}}}{d}\prod_{\substack{i_k=0\\ i_k\neq l}}^{j_k-1} p^{\vec{\eta}^{(n-1)}_{j_{n-1}+1}\circ\vec{\eta}^{(k)}_{i_k}}\left(\eta^{(k)}_{i_k},\eta^{(k)}_{i_k+1}\right). \tag{6.5}$$

Recall (4.6) and let $\chi_k^{(N)}$ be obtained by replacing $\Delta^{(k)}_{j_k+1}$ in (6.1) by

e:chireplace

$$d^{-1}I_{\{\eta^{(k)}_{l+1}-\eta^{(k)}_{l}=\pm e_r, r\le d_0^*\}}I_{\left\{(\eta^{(k)}_{j_k},\eta^{(k)}_{j_k+1})\in\{(\eta^{(k-1)}_{l},\eta^{(k-1)}_{l+1}):0\le l\le j_{i-1}\}\right\}} \tag{6.6}$$

and by bounding $\Delta^{(n)}_{j_n+1}$ for $n\neq k$ by $|\Delta^{(n)}_{j_n+1}|$.

Letting $\gamma^{(N)}=\sum_{k=1}^N\gamma_k^{(N)}$ and $\chi^{(N)}=\sum_{k=1}^N\chi_k^{(N)}$, we obtain that

e:3terms

$$\sum_m\sum_{x,y}|\varphi_m^{(N,1)}(x,y)|\le\rho^{(N)},\quad \sum_m\sum_{x,y}|\varphi_m^{(N,2)}(x,y)|\le\gamma^{(N)},\quad\text{and}\ \sum_m\sum_{x,y}|\varphi_m^{(N,3)}(x,y)|\le\chi^{(N)}. \tag{6.7}$$

lem:rhobound

Lemma 6.1 (Bounds on $\rho^{(N)}$). *We have* $\rho^{(1)}\le 2d_0^*d^{-2}\mathcal{E}_0(d_1,d_1d^{-1})$*, and, for* $N\ge 2$*,*

e:rhobound

$$\rho^{(N)}\le\frac{2d_0^*G_{d_1}(0)}{d^2d_1}\mathcal{E}_1(d_1,d_1d^{-1})\,(a_{d,d_1})^{(N-2)}. \tag{6.8}$$

Proof. For $N\ge 1$,

e:summrho

$$\begin{aligned}\rho^{(N)}=\frac{2d_0^*}{d}\sum_{\vec{j}\in\mathcal{A}_N}E^{\dagger\varnothing}\Bigg[&E^{\vec{\eta}^{(0)}_{j_0+1}}\Big[\sum_{\eta^{(1)}_{j_1+1}}|\Delta^{(1)}_{j_1+1}|\dots\\ &E^{\vec{\eta}^{(N-2)}_{j_{N-2}+1}}\Big[\sum_{\eta^{(N-1)}_{j_{N-1}+1}}|\Delta^{(N-1)}_{j_{N-1}+1}|E^{\vec{\eta}^{(N-1)}_{j_{N-1}+1}}\Big[\sum_{\eta^{(N)}_{j_N+1}}|\Delta^{(N)}_{j_N+1}|\Big]\Big]\dots\Big]\Bigg]\le\Pi_N(\vec{g}_N),\end{aligned} \tag{6.9}$$

where $g_{0,j_0}=2d_0^*d^{-1}I_{\{j_0=0\}}$, and $g_{i,j_i}=f_{i,j_i}$ for $i\ge 1$ (see (5.11)) and the $\vec{\eta}^{(i)}$ for $i\ge 1$ are RWpRE, while $\vec{\eta}^{(0)}$ is a 1-step simple random walk in the *first* d_0^* *coordinates only.* This latter difference is indicated by the dagger in the notation $E^{\dagger\varnothing}$. Since we have already established the relevant bounds on the f_{i,j_i} to complete the proof of Lemma 6.1 by applying Lemma 5.1, it is enough to establish that

$$\sum_{j_0=0}^{\infty}(j_0+1)E_0^{\dagger\vec{\eta}}[g_{0,j_0}]\le\frac{2d_0^*}{d},$$

which is trivial. □

lem:chibound

Lemma 6.2 (Bounds on $\chi^{(N)}$). *We have* $\chi^{(1)}\le d_0^*d^{-2}\mathcal{E}_0(d_1,d_1d^{-1})$*, and, for* $N\ge 2$*,*

$$\chi^{(N)}\le[d_0^*+(N-1)d_0]\frac{G_{d_1}(0)}{d^2d_1}\mathcal{E}_1(d_1,d_1d^{-1})\,(a_{d,d_1})^{(N-2)} \tag{6.10}$$

Proof. If $k > 1$ then set $\phi^{(k)}_{i,j_i} = f_{i,j_i}$ for all i. For $k = 1$ also define $\phi^{(k)}_{0,j_0}$ by replacing d_0 by d_0^* in the indicator function in the definition of f_{0,j_0}.

Observe that

$$\chi^{(N)}_k \leq \Pi_N(\vec{\phi}^{(k)}_N) \tag{6.11}$$

The claimed bound on $\chi^{(N)}$, is then easily obtained by applying Lemma 5.1 to each of the $\chi^{(N)}_k$ and summing over k. ☐

The following lemma can be proved as in Lemma 4.3 of [4], with trivial changes.

lem:togreens2

Lemma 6.3. *Let $P^{\leftrightarrow_l,\vec{\eta}_m}$ denote the law of a self-interacting random walk with history $\vec{\eta}_m$, such that the steps of the walk in d_1 coordinate directions are those of a simple random walk in d_1 dimensions (independent of the entire history of the walk) and such that the l^{th}-step is a simple random walk step in the first d_0^* coordinates only. Then, for $i \geq 0$,*

e:greensb2

$$\sum_{j=1}^{\infty} \frac{(j+i)!}{j!} \sum_{l=1}^{j} P^{\leftrightarrow_l,\vec{\eta}_m}(\omega_j = u) \leq (i+1)! \left(\frac{d}{d_1}\right)^{i+2} G^{*(i+2)}_{d_1}(0). \tag{6.12}$$

Define

e:epsilonddef

$$\epsilon(d,d_1,d_0^*) = \frac{2d_0 d_0^*}{d_1^2}\left[\frac{2dG^{*3}_{d_1}(o)G_{d_1}(o)}{d_1^2} + G^{*2}_{d_1}(0)\mathcal{E}_1(d_1,d_1d^{-1})\right]. \tag{6.13}$$

lem:gammabound

Lemma 6.4 (Bounds on $\gamma^{(N)}$)**.** *We have $\gamma^{(1)} \leq 2d_0d_0^*d_1^{-2}G^{*2}_{d_1}(0)$, $\gamma^{(2)} \leq \epsilon(d,d_1,d_0^*)$ and, for all $N \geq 3$,*

$$\gamma^{(N)} \leq \epsilon(d,d_1,d_0^*)(a_{d,d_1})^{N-2} + (N-2)\frac{4d_0d_0^*}{d_1^4}G_{d_1}(0)G^{*3}_{d_1}(0)\mathcal{E}_1(d,d_1d^{-1})\,(a_{d,d_1})^{N-3}. \tag{6.14}$$

Proof. We rewrite

e:pigamma

$$\begin{aligned}\gamma^{(N)}_k = \sum_{\vec{j}\in\mathcal{A}_N} E^{\varnothing}\Bigg[E^{\vec{\eta}^{(0)}_1}\Bigg[\sum_{\eta^{(1)}_{j_1+1}} |\Delta^{(1)}_{j_1+1}| \dots E^{\vec{\eta}^{(k-2)}_{j_{k-2}+1}}\Big[\sum_{\eta^{(k-1)}_{j_{k-1}+1}} |\Delta^{(k-1)}_{j_{k-1}+1}| \\ \sum_{l=1}^{j_k} E^{\leftrightarrow_l\vec{\eta}^{(k-1)}_{j_{k-1}+1}}\Big[\sum_{\eta^{(k)}_{j_k+1}} |\Delta^{(k)}_{j_k+1}| E^{\vec{\eta}^{(k)}_{j_k+1}}\Big[\sum_{\eta^{(k+1)}_{j_{k+1}+1}} |\Delta^{(k+1)}_{j_{k+1}+1}| \dots E^{\vec{\eta}^{(N-1)}_{j_{N-1}+1}}\Big[\sum_{\eta^{(N)}_{j_N+1}} |\Delta^{(N)}_{j_N+1}|\Big]\dots\Big]\dots\Big]\Big]\Bigg]\Bigg] \\ \leq \Pi^{(k)}_N(\vec{h}_N),\end{aligned} \tag{6.15}$$

where $h_{i,j_i} = f_{i,j_i}$ for $i \neq k$, $h_{k,j_k} = 2d_0^*d^{-1}f_{k,j_k}$, and the $\vec{\eta}^{(i)}$ for $i \neq k$ are RWpRE, while $\vec{\eta}^{(k)}$ is a RWpRE except that its lth step is a simple random walk step in the first d_0^* coordinates. This is indicated by the left-right arrow with subscript l in the notation $E^{\leftrightarrow_l\vec{\eta}^{(k-1)}_{j_{k-1}+1}}$.

When $N = 1$, then also $k = 1$ and we use (6.12) with $i = 0$, and Lemma 5.1 to get the required bound.

When $N > 1$ and $k = 1$ we use the same bounds as in the proof of Proposition 5.3 except that we have an extra factor $2d_0^*d^{-1}$ and use (6.12) with $i = 1$ on the term $k = 1$. This gives us a bound on $\gamma^{(N)}_1$ (when $N > 1$) of

e:kis1

$$\frac{4d_0d_0^*}{d}\left(\frac{d}{d_1}\right)^3 G^{*3}_{d_1}(o)\frac{G_{d_1}(o)}{dd_1}\,(a_{d,d_1})^{N-2} = \frac{4d_0d_0^*d}{d_1^4}G^{*3}_{d_1}(o)G_{d_1}(o)\,(a_{d,d_1})^{N-2}. \tag{6.16}$$

When $N > 1$ and $k = N$, we use the same bounds as in the proof of Proposition prp:pibound 5.3 except that we have an extra factor $2d_0^*d^{-1}$ and use e:greensb2 (6.12) with $i = 0$ on the term $k = N$. This gives us a bound on $\gamma_N^{(N)}$ (when $N > 1$) of

e:kisN

$$\frac{2d_0d_0^*}{dd_1^2}G_{d_1}^{*2}(0)\mathcal{E}_1(d, d_1d^{-1})(a_{d,d_1})^{N-2}. \tag{6.17}$$

Similarly when $N > 1$ and $1 \neq k \neq N$ (so $N > 2$), we use e:greensb2 (6.12) with $i = 1$ on term k to get a bound on $\gamma_k^{(N)}$ of

$$\frac{4d_0d_0^*}{d_1d^3}G_{d_1}(0)\mathcal{E}_1(d, d_1d^{-1})\left(a_{d,d_1}\right)^{N-3}\left(\frac{d}{d_1}\right)^3 G_{d_1}^{*3}(0) = \frac{4d_0d_0^*}{d_1^4}G_{d_1}(0)G_{d_1}^{*3}(0)\mathcal{E}_1(d, d_1d^{-1})\left(a_{d,d_1}\right)^{N-3}. \tag{6.18}$$

e:kisN2

Summing these expressions over k completes the proof of the lemma. □

cor:bounds

Corollary 6.5 (Summary of bounds)**.** *For all d, d_1 such that $a_{d,d_1} < 1$,*

e:RHO

$$\frac{d}{2}\sum_{N=1}^{\infty}\rho^{(N)} \leq \frac{d_0^*\mathcal{E}_0(d_1, d_1d^{-1})}{d} + \frac{d_0^*G_{d_1}(o)\mathcal{E}_1(d_1, d_1d^{-1})}{dd_1(1-a_{d,d_1})} \tag{6.19}$$

e:CHI

$$\frac{d}{2}\sum_{N=1}^{\infty}\chi^{(N)} \leq \frac{d_0^*\mathcal{E}_0(d_1, d_1d^{-1})}{2d} + \frac{G_{d_1}(o)\mathcal{E}_1(d_1, d_1d^{-1})}{2dd_1(1-a_{d,d_1})}\left[d_0^* + \frac{d_0}{1-a_{d,d_1}}\right] \tag{6.20}$$

e:GAMMA

$$\frac{d}{2}\sum_{N=1}^{\infty}\gamma^{(N)} \leq \frac{dd_0d_0^*G_{d_1}^{*2}(o)}{d_1^2} + \frac{d\epsilon(d, d_1, d_0^*)}{2(1-a_{d,d_1})} + \frac{dG_{d_1}(0)G_{d_1}^{*3}(0)}{(1-a_{d,d_1})^2}\frac{2d_0d_0^*}{d_1^4}\mathcal{E}_1(d, d_1d^{-1}). \tag{6.21}$$

Proof. The results are easily obtained by summing each of the bounds in Lemmas lem:rhobound 6.1, lem:chibound 6.2 and lem:gammabound 6.4 over N. □

thm:main

7 Proof of Theorem 2.1

sec-proof

By Lemma B.3 of HS92b [2], $G_{d_1}^{*(i)}(o)$ is decreasing in d_1 for each i and thus so is a_{d,d_1}. As in Section 5 of HH08mono [4],

$$\mathcal{E}_i(d_1, d_1d^{-1}) = \left(\frac{d}{d_1}\right)^{i+1} G_{d_1}^{*(i+1)}(o) - 1, \tag{7.1}$$

which is decreasing in d_1 (recall that $d = d_0 + d_1$) for fixed d_0. It follows that $\epsilon(d, d_1, d_0^*)$ is also decreasing in d_1 and that all of the bounds in Corollary cor:bounds 6.5 (and e:pibound (5.10)) are decreasing in d_1, and converge to 0 as $d_1 \to \infty$ for any d_0, d_0^*. This completes the proof of the theorem. □

8 Explicit values of d_c

sec:table

Table tab:speedform 1 was created by fixing $d_1 \geq 5$ and finding the largest d_0 for which the numerical upper bound on a_{d,d_1} is less than 1, and hence the speed formula (e:theta (4.1) and e:theta2 (4.7)) is valid for those values of d_0, d_0^*, d. Note that d_0^* is not relevant for our bound on a_{d,d_1}.

d_0	d_0^*	d_1
7	≤ 7	5
17	≤ 17	6
28	≤ 28	7

`tab:speedform`
Table 1: Values of d_0 up to which the speed formula $\overset{\texttt{e:theta}}{(4.1)}$ is known to be valid for $d_1 = 5, 6, 7$.

d_0	1	2	2	3	3	3	4	5
d_0^*	1	1	2	1	2	3	1	1
d_c	7	9	12	12	16	20	14	17

`tab:mono`
Table 2: Values of d_c for which Theorem $\overset{\texttt{thm:main}}{2.1}$ holds for various combinations of d_0 and $d_0^* \le d_0$.

Table $\overset{\texttt{tab:mono}}{2}$ was created by fixing d_0 and $d_0^* \le d_0$ and finding the smallest value of d_1 for which the numerical upper bound on $\overset{\texttt{e:RHO}}{(6.19)}+\overset{\texttt{e:CHI}}{(6.20)}+\overset{\texttt{e:GAMMA}}{(6.21)}$ is less than 1, proving that monotonicity in β holds for those values of d_0, d_0^*, d. Note that we require $d_1 \ge 7$ for $G^{*3}(o)$ to be finite.

To establish the results in both tables we have used rigorous upper bounds on $G_{d_1}^{*i}(o)$ for $i = 1, 2, 3$, generously provided by Takashi Hara. Upper bounds are sufficient since all quantities of interest are increasing functions of the $G_{d_1}^{*i}(o)$.

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