Degenerate random environments.

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May 3, 2010

Abstract

We consider connectivity properties of certain i.i.d. random environments on $\mathbb{Z}^d$, where at each location some steps may not be available. Site percolation and oriented percolation can be viewed as special cases of the models we consider. In such models, one of the quantities most often studied is the (random) set of vertices that can be reached from the origin by following a connected path. More generally, for the models we consider, multiple different types of connectivity are of interest, including: the set of vertices that can be reached from the origin; the set of vertices from which the origin can be reached; the intersection of the two. As with percolation models, many of the models we consider admit, or are expected to admit phase transitions. Among the main results of the paper is a proof of the existence of phase transitions for some two-dimensional models that are non-monotone in their underlying parameter, and an improved bound on the critical value for oriented site percolation on the triangular lattice. The connectivity of the random directed graphs provides a foundation for understanding the asymptotic properties of random walks in these random environments, which we study in a second paper.

1 Introduction

We start with an elementary example, to illustrate the kind of questions we will be asking.

Perform site percolation with parameter p on the lattice $\mathbb{Z}^2$. From each occupied vertex $x = (x^{[1]}, x^{[2]})$, insert two directed edges, one pointing up $\uparrow$ and one pointing right $\rightarrow$. If x is not occupied, insert directed edges pointing up $\uparrow$ and left $\leftarrow$ (see Figure 1). In the resulting random directed graph, made up of configurations $\uparrow\!\rightarrow$ and $\leftarrow\!\uparrow$, there is of course an arrow pointing up from every vertex, so in particular from any vertex x the set of vertices $\mathcal{C}_x$ that can be reached from x is infinite. Likewise for any y, the set of verticies $\mathcal{B}_y = \{x : y \in \mathcal{C}_x\}$ from which y can be reached is also infinite. However, for each x, $\mathcal{M}_x := \mathcal{C}_x \cap \mathcal{B}_x$ is finite.

The random walk X_n (choosing uniformly among available steps) in such a random environment is then transient, but in this particular case much more can be said. Since at each step the random

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†Department of Mathematics and Statistics, York University. E-mail salt@yorku.ca; Salisbury's research is supported in part by NSERC. Part of this work was carried out during a visit to the University of Auckland, whose hospitality he gratefully acknowledges.

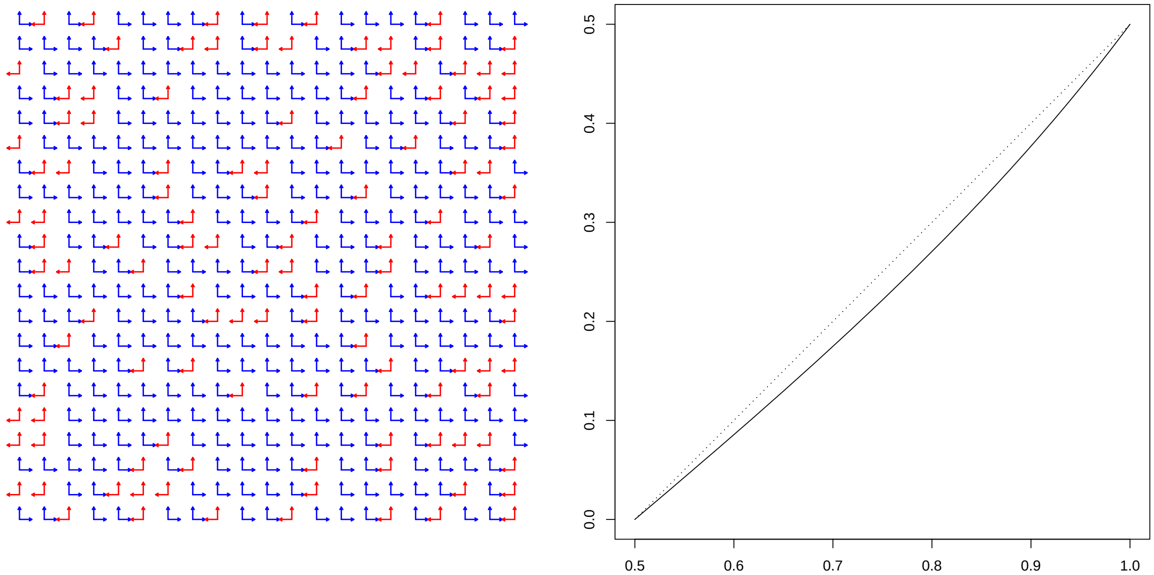

Figure 1: A finite region of a degenerate environment in two dimensions such that $\mu(\{\uparrow, \rightarrow\}) = p = .75$, $\mu(\{\leftarrow, \uparrow\}) = 1 - p = .25$, and the first coordinate of the velocity as a function of p.

walk has probability 1/2 of moving up and probability 0 of moving down, the random walk in this random environment trivially has a limiting velocity in the vertical direction given by $v^{[2]} = 1/2$. In addition, each upward step constitutes a renewal for the walk since the the environment seen thereafter has no intersection with the past. For $n \geq 0$, let $\tau_n = \inf\{m \geq 0 : X_m^{[2]} = n\}$. Then for $i \geq 1$, $T_i = \tau_i - \tau_{i-1}$ are i.i.d. Geometric(1/2) random variables (with mean 2), and $Y_i = X_{\tau_i - 1}^{[1]} - X_{\tau_{i-1}}^{[1]}$ are i.i.d. random variables, independent of the $\{T_i\}_{i \geq 1}$. Let $N_n = \sup\{m \geq 0 : \tau_m \leq n\}$. Then almost surely,

$$\frac{X_n^{[1]}}{n} = \frac{\sum_{i=1}^{N_n} Y_i + \sum_{i=\tau_{N_n}+1}^{n} (X_i^{[1]} - X_{i-1}^{[1]})}{n} = \frac{N_n}{n} \frac{\sum_{i=1}^{N_n} Y_i}{N_n} + \frac{\sum_{i=\tau_{N_n}+1}^{n} (X_i^{[1]} - X_{i-1}^{[1]})}{n} \to \frac{E[Y_1]}{E[T_1]},$$

as $n \to \infty$, where we have used the fact that $|\sum_{i=\tau_{N_n}+1}^{n} (X_i^{[1]} - X_{i-1}^{[1]})| \leq T_{N_{n+1}}$. The expectations can be calculated explicitly to get (see [10] and also Figure 1)

$$v^{[1]} = \frac{(2p-1)(p^2 - p + 6)}{6(2-p)(1+p)}.$$

Compare this with the speed $\tilde{v}^{[1]} = p - \frac{1}{2}$ of a true random walk that goes up with probability 1/2, right with probability $p/2$, and left with probability $(1-p)/2$, we see that the speeds agree for $p = 0, 1/2$, and 1, but the RWRE is slower in between.

The above random environment is degenerate in the sense that some edges are missing. The "uniform ellipticity" condition that is normally assumed in studying RWRE, fails in this case. The objective of this paper is to study the connectivity structure of such random environments, but in a more general setting, where the above simple reasoning may not apply. In a second paper [10] we consider also asymptotic properties of random walks (that choose uniformly from available steps) in such random environments, including speed calculations and transience results, where possible.

This paper is organised as follows. In Section 1.1 we introduce the random environments that are the main objects of study of this paper, as well as various notions of connectivity in the random graphs. We describe the main results of the paper in Section 1.2. In Sections 2, 3 and 4 we examine three of these notions of connectivity in such random directed graphs, with the main results being proved in Sections 3 and 4.

1.1 The model

For fixed $d \geq 2$ let $\mathcal{E} = \{\pm e_i : i = 1, \ldots, d\}$ be the set of unit vectors in $\mathbb{Z}^d$, and let $\mathcal{P}$ denote the power set of $\mathcal{E}$. For any set A, let $|A|$ denote the cardinality of A. Let μ be a probability measure on $\mathcal{P}$. A *degenerate random environment* is an element $\mathcal{G} = \{\mathcal{G}_x\}_{x\in\mathbb{Z}^d}$ of $\mathcal{P}^{\mathbb{Z}^d}$, equipped with the product σ-algebra and the product measure $\nu = \mu^{\otimes\mathbb{Z}^d}$. We denote the expectation of a random variable Z with respect to ν by $\mathbb{E}[Z]$.

We say that the environment is 2-*valued* when μ charges exactly two points, i.e. there exist distinct $A_1, A_2 \subset \mathcal{P}$ and $p \in (0, 1)$ such that $\mu(\{A_1\}) = p$ and $\mu(\{A_2\}) = 1 - p$.

Example 1.1. ($\uparrow\!\rightarrow\ \leftarrow\!\uparrow$)*: Let* $A_1 = \{\uparrow, \rightarrow\}$ *and* $A_2 = \{\uparrow, \leftarrow\}$*. Setting* $\mu(\{A_1\}) = p$, $\mu(\{A_2\}) = 1-p$ *we obtain the example discussed in the introduction (see also Figure 1).*

Example 1.2 (Site percolation: $\leftrightarrow\!\!\!\updownarrow\ \cdot$)**.** *Let* $\mu(\{\mathcal{E}\}) = p$ *and* $\mu(\{\emptyset\}) = 1 - p$*. Then the random environment induced by* μ *is simply site percolation. More precisely, the site percolation clusters will agree with the equivalence classes under the communication relation, as defined below.*

Example 1.3 (Oriented site percolation: $\uparrow\!\rightarrow\ \cdot$)**.** *Let* $\mu(\{\uparrow, \rightarrow\}) = p$ *and* $\mu(\{\emptyset\}) = 1 - p$*. Then the random environment induced by* μ *is simply oriented site percolation.*

Example 1.4 (Oriented bond percolation)**.** *Let* $A = \{+e_i : i = 1, \ldots, d\}$ *and fix* $p \in (0, 1)$*. Define the measure* μ *by* $\mu(\{B\}) = 0$ *if* $B \cap A^c \neq \emptyset$ *and* $\mu(\{B\}) = p^{|B|}(1 - p)^{d-|B|}$ *for* $B \subset A$*. Then the random environment induced by* μ *is the oriented (bond) percolation model of [8, Section 12.8].*

Example 1.5. ($\uparrow\rightarrow$)*: If* $\mu(\{\uparrow\}) = p$ *and* $\mu(\{\rightarrow\}) = 1 - p$*, we produce a web of coalescing random walks, as in Arratia [1] or Toth-Werner [21].*

Example 1.6 (Randomly oriented lattices:)**.** *In* $\mathbb{Z}^2$*, re-label the four unit vectors in* $\mathcal{E}$ *as* $f_1, \ldots, f_4$*. For* $0 < \lambda_1, \ldots, \lambda_4 < 1$ *take* $\mu(A) = \prod_{i=1}^4 \lambda_i^{1_A(f_i)}(1-\lambda_i)^{1_{A^c}(f_i)}$*. Then each bond is randomly oriented in one or both directions. In other words, this gives Grimmett's "Independent randomly oriented lattice" model. See Grimmett [9], Wu and Zuo [24], and Linusson [15]. This includes "Diode-resistor percolation" as a special case. See Dhar et al [5], Redner [19], and Wierman [22]*

This paper will be organized thematically, in the sense that each section treats a class of results, using related methods. Multiple specific models (primarily 2-dimensional 2-valued environments) are treated in each section, and at the end of the paper we will summarize, for a list of 2-dimensional 2-valued models, what is known for each one.

Definition 1.7. *Given an environment $\mathcal{G}$:*

- *We say that x is* connected *to y, and write $x \to y$ if: there exists an $n \geq 0$ and a sequence $x = x_0\, x_1, \ldots, x_n = y$ such that $x_{i+1} - x_i \in \mathcal{G}_{x_i}$ for $i = 0, \ldots, n-1$. Let $\mathcal{C}_x = \{y \in \mathbb{Z}^d : x \to y\}$, and $\mathcal{B}_y = \{x \in \mathbb{Z}^d : x \to y\}$.*

- *We say that x and y are* mutually connected*, or that they* communicate*, and write $x \leftrightarrow y$ if $x \to y$ and $y \to x$. Let $\mathcal{M}_x = \{y \in \mathbb{Z}^d : x \leftrightarrow y\} = \mathcal{B}_x \cap \mathcal{C}_x$.*

- *A set of vertices $F \subset \mathbb{Z}^d$ is said to be* absorbing *if no vertex in F is connected to any vertex in F^c. Any such set can be written as $F = \cup_{x \in F} \mathcal{C}_x$. Note that for every x, $\mathcal{C}_x$ is an absorbing set.*

- *Given an environment $\mathcal{G}$, we say that a nearest neighbour path in $\mathbb{Z}^d$ is* admissible in $\mathcal{G}$ *if that path consists of edges in $\mathcal{G}$.*

The list of specific models that we will consider is fairly long, but to fix ideas, we close this section by describing three of the most interesting 2-dimensional, 2-valued environments.

Example 1.8. ($\uparrow\!\!\!\rightarrow$ $\leftarrow\!\!\!\downarrow$)*: Let $A_1 = \{\uparrow, \rightarrow\}$ and $A_2 = \{\downarrow, \leftarrow\}$, and take $\mu(\{A_1\}) = p$, $\mu(\{A_2\}) = 1-p$. We'll show that for $p \approx 0$ or 1, all $\mathcal{M}_x$ are finite. We'll also show that for $p \approx 1/2$ there is a giant component for the directed graph. In other words, there exists an x such that $\mathcal{M}_x$ is infinite, and moreover, all other $\mathcal{M}_y$ either agree with it or are finite. There are two phase transitions in the model as p varies – with $\mathcal{M}_x$ moving from being finite, to infinite, and back to finite. The model is not monotone, in the sense that changing the local environment can both open and close connections. Nevertheless we'll relate the critical p's to critical values for monotone percolation models. See the following: Figure 2, Figure 6, Theorem 3.14, Corollary 4.3, and Theorem 4.16.*

Note that this model has superficial resemblances to corner percolation (see Pete [18]), and to the Lorentz lattice gas model (see §13.3 of Grimmett [8]), though those models in fact seem unrelated.

Example 1.9. ($\leftrightarrow\updownarrow$)*: Let $A_1 = \{\rightarrow, \leftarrow\}$ and $A_2 = \{\uparrow, \downarrow\}$, and take $\mu(\{A_1\}) = p$, $\mu(\{A_2\}) = 1-p$. This model is a degenerate version of the "good-node bad-node" model of Lawler [13]. We'll show that there is no phase transition of the above type for this model. In other words, as long as $0 < p < 1$ there is a giant component for the directed graph, in the sense given above. See Corollary 4.12.*

Example 1.10. ($\leftarrow\!\downarrow\!\rightarrow$ $\uparrow$)*: $\mu(\{\rightarrow, \downarrow, \leftarrow\}) = p$ and $\mu(\{\uparrow\}) = 1 - p$. We'll show that for $p \approx 0$ all $\mathcal{M}_x$ are finite, while for $p \approx 1$ there is a giant component for the directed graph, in the sense given above. The model is not monotone, but still has a single phase transition, which we will connect to that of a monotone percolation model. See Figure 7, Theorem 3.15, Corollary 4.3, and Theorem 4.15.*

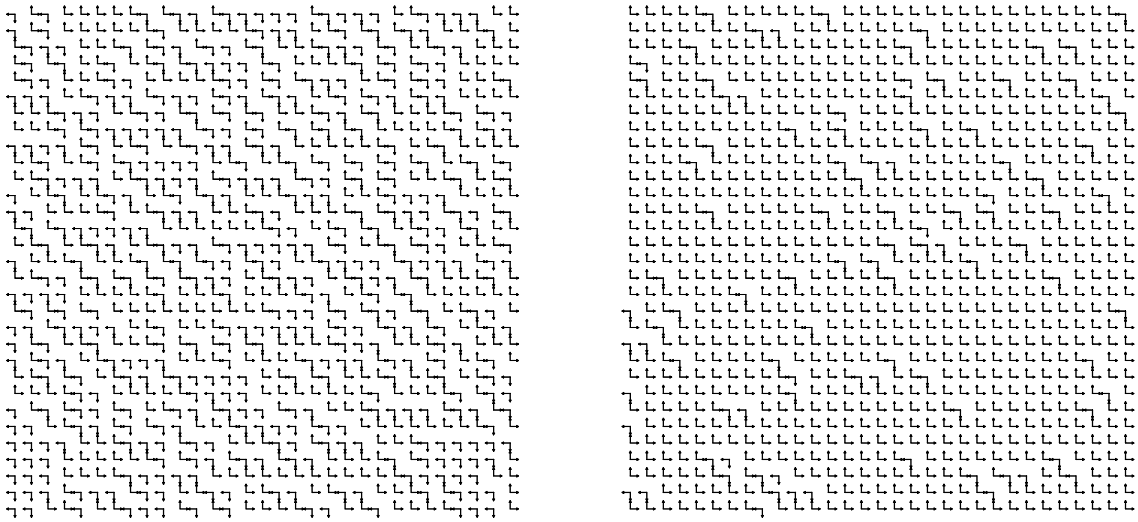

Figure 2: Finite regions of the random environment in Example 1.8 for $p = .5$ and $p = .9$ respectively.

1.2 Main results

Since we study a whole class of models in this paper, there are numerous results. Many are short and elementary, while some are substantial. As a guide to the reader, we summarize the most significant contributions here, and then at the end of the paper give a comparison of the main models considered.

We use a broad range of classical methods that have been successful in studying other percolation models, including blocking configurations, duality results and self-avoiding path counting arguments. Many of our models in fact form 1-parameter families that have sharp phase transitions. We believe these are deserving of further attention. A key feature of these models is that they are not monotone – when coupled together, increasing the parameter p can both open new connections and close off old ones. It is this feature that makes the sharpness of phase transitions non-trivial. Beyond percolation-type considerations, the results we obtain give the foundation for studying random walks in these (non-elliptic) random environments. That study will appear in future work.

In many of our models, we exploit a tool that is not present in standard percolation models, namely the existence of multiple subnetworks of coalescing random walks. Being able to get one's hands on these random walk paths is the key factor in getting many of our arguments to work.

In the authors' opinion, the following are the highlights of this paper:

1. Introducing a new and interesting class of random directed graph models (Section 1.1).

2. Classifying all the 2-valued 2-dimensional cases (culminating in Table 5), and identifying models (see Examples 1.8 and 1.10) with interesting phase transitions.

3. Proving a structure theorem (see Proposition 3.10 and Corollary 3.12), giving the possible forms of $\mathcal{B}_x$, under fairly broad conditions.

4. Proving the existence of sharp phase transitions, both for infinite $\mathcal{B}_x$ clusters (e.g. see Theorems 3.14 and 3.15) and for infinite $\mathcal{M}_x$ clusters (see Corollary 4.11). As noted earlier, it is not trivial that sharp transitions should exist, because the models are not monotone.

5. Proving that the clusters $\mathcal{M}_x$ of Example 1.8 may be infinite for p in a neighbourhood of $\frac{1}{2}$ (see Theorem 3.14 and Corollary 4.11). Prior to carrying out the analysis, it seemed equally plausible to the authors that the clusters would be finite but grow as $p \to \frac{1}{2}$, and only be infinite for $p = \frac{1}{2}$.

6. Improving existing rigorous bounds on the critical values of oriented site-percolation models on the triangular lattice. These follow from bounds on the critical values for our random directed graph models, together with a duality argument (see Theorems 4.15 and 4.16).

2 The set of points $\mathcal{C}_x$ that can be reached from x

In this section we investigate properties of the random sets $\mathcal{C}_x \subset \mathbb{Z}^d$. It is interesting to investigate conditions on μ which ensure that $0 < \nu(|\mathcal{C}_o| = \infty) < 1$, and we will return to this briefly in Section 2.1. For now, the reader can easily check that the following are some of the many environments in which $\mathcal{C}_o$ can be finite with positive probability.

Example 2.1. (↑↳↓↱↰↓↲↑)*:* $\mu(\{\uparrow, \rightarrow\}) > 0$*,* $\mu(\{\downarrow, \rightarrow\}) > 0$*,* $\mu(\{\leftarrow, \downarrow\}) > 0$*,* $\mu(\{\uparrow, \leftarrow\}) > 0$*. Then* $\nu(|\mathcal{C}_o| = 4) > 0$*.*

Example 2.2. $(\uparrow\downarrow)$*:* $\mu(\{\uparrow\}) > 0$ *and* $\mu(\{\downarrow\}) > 0$*. Then* $\nu(|\mathcal{C}_o| = 2) > 0$*.*

Example 2.3. *Any model having* $\mu(\{\emptyset\}) > 0$ *has* $\nu(|\mathcal{C}_o| = 1) > 0$*.*

As for standard percolation models, when $\mathcal{C}_o$ is infinite with positive probability, there is almost surely an infinite $\mathcal{C}_x$.

Lemma 2.4. *If* $\nu(|\mathcal{C}_o| = \infty) > 0$ *then* $\nu(\exists x : |\mathcal{C}_x| = \infty) = 1$*.*

Proof. Let $\mathcal{G}$ be a degenerate random environment on $\mathbb{Z}^d$. On any subgraph G_i of $\mathbb{Z}^d$ we have that for any $x \in G_i$,

$$\{|\mathcal{C}_x| = \infty \text{ in } G_i\} = \cup_{y \in G_i} \left(\{y - x \in \mathcal{G}_x\} \cap \{|\mathcal{C}_y| = \infty \text{ in } G_{i+1} = G_i \setminus x\}\right).$$

Starting with $G_0 = \mathbb{Z}^d$ and using this fact inductively on deleted graphs $G_{i+1} = G_i \setminus x_i$ we obtain

$$\{|\mathcal{C}_x| = \infty\} = \{\exists \infty - \text{self-avoiding path } x = x_0, x_1, \cdots : x_{i+1} - x_i \in \mathcal{G}_{x_i} \forall i \geq 0\}.$$

This implies that

$$\{\exists x : |\mathcal{C}_x| = \infty\} = \{\exists \infty - \text{self-avoiding path } x_0, x_1, \cdots : x_{i+1} - x_i \in \mathcal{G}_{x_i} \forall i \geq 0\}.$$

By pruning off initial segments of such paths, this event equals

$$\{\forall n\, \exists \infty - \text{self-avoiding path } x_0, x_1, \cdots : x_{i+1} - x_i \in \mathcal{G}_{x_i}, |x_i| > n\, \forall i \geq 0\}.$$

The event on the right clearly does not depend on any finite collection of the $\mathcal{G}_y$, hence it is a tail event. The result follows since $\nu(\exists x : |\mathcal{C}_x| = \infty) \geq \nu(|\mathcal{C}_o| = \infty) > 0$. □

When studying degenerate random environments, and random walks therein, our principal interest will be in situations where the following condition holds:

$$\boxed{\nu(|\mathcal{C}_o| < \infty) = 0}. \tag{2.1}$$

Since $\nu\Big(\cup_{x\in\mathbb{Z}^d} \{|\mathcal{C}_x| < \infty\}\Big) \leq \sum_{x\in\mathbb{Z}^d} \nu(|\mathcal{C}_x| < \infty) = \sum_{x\in\mathbb{Z}^d} \nu(|\mathcal{C}_o| < \infty)$, (2.1) prevents the random walk from getting stuck on a finite set of sites. In many cases this will be immediately obvious from the following result:

Lemma 2.5. *Fix $d \geq 2$. If there exists an orthogonal set V of unit vectors such that $\mu(\{A : A \cap V \neq \emptyset\}) = 1$ then $|\mathcal{C}_o| = \infty$, ν-almost surely.*

Proof. Construct an infinite (self-avoiding) path by always following a vector chosen from V. □

The above examples are nearly exhaustive in the 2-dimensional, 2-valued setting, as the following shows:

Corollary 2.6. *Suppose $d = 2$ and μ is 2-valued. Then $\mathcal{C}_o$ is infinite ν-almost surely, except for the models of Examples 2.2 and 2.3 (and the 90° rotation of the former).*

Proof. Let A_1 and A_2 be distinct, $\mu(\{A_i\}) > 0$. Assume that $\mathcal{C}_o$ is finite with positive probability. Without loss of generality, $\rightarrow \in A_1$. Then Lemma 2.5 shows that either $A_2 = \emptyset$ or $A_2 = \{\leftarrow\}$. If $A_2 = \{\leftarrow\}$, it follows further that $A_1 = \{\rightarrow\}$. □

In models where the $\mathcal{C}_x$ are infinite almost surely, we can go on to ask about intersections of these classes (or equivalently, the coalescence of lines of descent). These type of models include the web of coalescing random walks, as in Arratia [1] or Toth-Werner [21]. In that context, the following argument, though familiar, will be useful later on.

Lemma 2.7. *Consider the model ($\uparrow\rightarrow$) of Example 1.5. Then for every x, y there is a z such that $x \rightarrow z$ and $y \rightarrow z$. In other words, $\mathcal{C}_x \cap \mathcal{C}_y \neq \emptyset$, μ-almost surely.*

Proof. Assume first that x and y both belong to the line $\{(i, j) : i+j = 0\}$. Follow the unique path from x (resp. y), and after n steps let X_n (resp. Y_n) be the first coordinate of the point reached. Then X_n and Y_n follow independent random walks (up to the time they coalesce), with probability p of standing in place, and probability $1-p$ of moving a step to the right. So $X_n - Y_n$ is a random walk, absorbed at 0, which moves $+1$ or -1 with probability $p(1-p)$ each, and otherwise stands in place. Since this nearest neighbour RW is symmetric, it hits 0 with probability 1, which is the desired conclusion.

If $x^{[1]} + x^{[2]} \neq y^{[1]} + y^{[2]}$, just follow the path from one point till it reaches the diagonal line the other starts on, and then apply the same argument. □

A model in which lines of descent do not intersect is:

Figure 3: Part of a realisation of the set C_o for (a) the model with $\mu(\{\uparrow,\rightarrow\}) = .8 = 1-\mu(\cdot)$, and (b) the model with $\mu(\{\uparrow,\rightarrow\}) = .8 = 1-\mu(\{\downarrow,\leftarrow\})$

Example 2.8. $(\leftrightarrow\rightarrow)$*:* $\mu(\{\leftarrow,\rightarrow\}) = p$ *and* $\mu(\{\rightarrow\}) = 1-p$, $0 < p < 1$.

But if we rule out this example, then in fact the behaviour of Lemma 2.7 is typical:

Corollary 2.9. *Suppose* $d = 2$ *and that* μ *is 2-valued. Then* $\mathcal{C}_x \cap \mathcal{C}_y \neq \emptyset$*,* ν*-almost surely, for every* x *and* y*, except in the following cases:*

(a) Models with $\nu(|\mathcal{C}_o| < \infty) > 0$ *(see Cor. 2.6)*

(b) The model $(\leftrightarrow\rightarrow)$ *of Example 2.8 and its rotations.*

Proof. We show that in every other case, the network contains that for $(\uparrow\rightarrow)$ or a rotation of it. In other words, that in every other case we can follow a path that uses only two of the four directions, and from arbitrary initial points, these paths will coalesce.

So suppose $A_1 \neq A_2$ and that $\mu(\{A_i\}) > 0$. Without loss of generality, neither is empty, and $\rightarrow \in A_1$. If $\uparrow \in A_2$ or $\downarrow \in A_2$ then Lemma 2.7 applies. If not then A_2 must be either $\{\leftarrow\}$ or $\{\leftarrow,\rightarrow\}$ or $\{\rightarrow\}$. The same is true for A_1, giving either case (b) above, or one of the models covered by Cor. 2.6. ∎

2.1 Percolation

We know that the model $(\uparrow\!\!\rightarrow\cdot)$ is oriented site percolation (Example 1.3), and that the model $(\leftrightarrow\!\!\updownarrow\cdot)$ is site percolation (Example 1.2). We denote the critical p's for these models by $1 > p_c^{\uparrow\rightarrow} > p_c^{\leftrightarrow\updownarrow} > 0$. In other words, for the model $(\uparrow\!\!\rightarrow\cdot)$, $|\mathcal{C}_o| = \infty$, with positive probability if $p > p_c^{\uparrow\rightarrow}$, and with probability 0 if $p < p_c^{\uparrow\rightarrow}$. See e.g. Figure 3. Similarly in the model $(\leftrightarrow\!\!\updownarrow\cdot)$ we have $|\mathcal{C}_o| = \infty$ with positive probability, if and only if $p > p_c^{\leftrightarrow\updownarrow}$. We denote by $p_c^{\mathrm{OBP}} \geq p_c^{\uparrow\rightarrow}$ the critical p for oriented bond percolation. These models have been well studied: see Grimmett [8] and Durrett [6].

Intermediate between the above two site percolation models is the following model.

Example 2.10. ($\overleftrightarrow{\downarrow}\cdot$)*: Partially-oriented site percolation. Here* $\mu(\{\leftarrow,\downarrow,\rightarrow\}) = p$ *and* $\mu(\{\emptyset\}) = 1-p$.

The model is monotone, so $\nu_p(|\mathcal{C}_o| = \infty)$ is increasing in p. Thus there is a critical value $p_c^{\overleftrightarrow{\downarrow}} \in [p_c^{\leftrightarrow\!\updownarrow}, p_c^{\uparrow\rightarrow}]$ such that $|\mathcal{C}_o| = \infty$ with positive probability if $p > p_c^{\overleftrightarrow{\downarrow}}$, and each $|\mathcal{C}_o| < \infty$ almost surely when $p < p_c^{\overleftrightarrow{\downarrow}}$. We do not know what happens at the critical value. Clearly also $|\mathcal{M}_x| < \infty$ since $\mathcal{M}_x$ is contained in the horizontal line containing x, and even $\mathcal{C}_x$ contains only finitely many points on that line. The model appears in Hughes [11], and Mártin and Vannimenus [16] give the estimate $p_c^{\overleftrightarrow{\downarrow}} \approx 0.6317$. We have not found rigorous bounds in the literature. From bounds on other models, one has

$$0.5416 \le p_c^{\overleftrightarrow{\downarrow}} \le 0.7491$$

based on $p_c^{\leftrightarrow\!\updownarrow} \ge 0.5416$ (due to Men'shikov and Pelikh [17]) and $p_c^{\uparrow\rightarrow} \le 0.7491$ (due to Balister, Bollobás and Stacey [2], improving on the bound 0.75 of Liggett [14]). The following result improves the lower bound.

Proposition 2.11. *In the partially-oriented percolation model* ($\overleftrightarrow{\downarrow}\cdot$) *of Example 2.10,* $\nu(|\mathcal{C}_o| < \infty) = 1$ *if* $p^3 - p^2 + 2p - 1 \le 0$*. This implies that* $p_c^{\overleftrightarrow{\downarrow}} \ge 0.56984$.

Proof. If $\mathcal{C}_o$ is infinite, then pruning off the vertices x of $\mathcal{C}_o$ for which $\mathcal{G}_x = \emptyset$ still leaves an infinite cluster. So let $M_n = \{y \in \mathcal{C}_o : y^{[2]} = -n, \mathcal{G}_y \neq \emptyset\}$. We are interested in whether or not M_n is empty or not, for large n. We can think of obtaining M_n in three steps. First we take points for which there is a $\downarrow$ connection from a point in M_{n-1}. We then close up M_n using the available $\leftarrow$ or $\rightarrow$ connections. Finally we delete points whose local environment is $\emptyset$.

Viewed that way, it is clear that the following construction builds a set of points M_n^0 that is at least as big as M_n. First put into M_n^0 every point that can be reached from M_{n-1}^0 using an allowed $\downarrow$. Second, close up all the gaps (whether allowed or not). Third, add in any points at either end that can be reached from this set using allowed $\leftarrow$ or $\rightarrow$. Fourth, take away points at either end as long as their local environment is $\emptyset$. This defines $M_n^0 \supset M_n$ consisting of all lattice points in $[L_n, U_n] \times \{-n\}$, where L_n and U_n are the inf and sup of the projection of M_n^0 on the 1st coordinate axis. The processes L_n and U_n should behave like independent random walks when they are far away from each other, so we can hope to decide whether or not they will ever cross. To make this precise we will use a coupling argument, of a type that will recur several times in this paper.

Let $l \le u$ and suppose that $L_n = l, U_n = u$. If $u' \ge u$ then we have $U_{n+1} = u'$ if $\mathcal{G}_{(j,-(n+1))} = \overleftrightarrow{\downarrow}$ for $u \le j \le u'$, while $\mathcal{G}_{(u'+1,-(n+1))} = \emptyset$. If $u > u' \ge l$ then we have $U_{n+1} = u'$ if $\mathcal{G}_{(j,-(n+1))} = \emptyset$ for $u \ge j > u'$, while $\mathcal{G}_{(u',-(n+1))} = \overleftrightarrow{\downarrow}$. The only other possible value of U_{n+1} is $-\infty$. Thus

$$\nu(U_{n+1} = u' \mid L_n = l, U_n = u) = \begin{cases} p^{u'-u+1}(1-p), & u' \ge u \\ (1-p)^{u-u'}p, & l \le u' < u. \end{cases}$$

By a similar calculation (or by symmetry),

$$\nu(L_{n+1} = l' \mid L_n = l, U_n = u) = \begin{cases} p^{l-l'+1}(1-p), & l' \le l \\ (1-p)^{l'-l}p, & l < l' \le u. \end{cases}$$

Now take independent sequences U_n^0 and L_n^0, with laws

$$\nu(U_{n+1}^0 = u' \mid U_n^0 = u) = \begin{cases} p^{u'-u+1}(1-p), & u' \geq u \\ (1-p)^{u-u'}p, & u' < u. \end{cases}$$

$$\nu(L_{n+1}^0 = l' \mid L_n^0 = l) = \begin{cases} p^{l-l'+1}(1-p), & l' \leq l \\ (1-p)^{l'-l}p, & l' > l. \end{cases}$$

As long as $\max(l, l') < \min(u, u')$ we will look at disjoint (and hence independent) environments when evaluating L_{n+1} and U_{n+1}. We conclude that for such values,

$$\nu(L_{n+1}^0 = l', U_{n+1}^0 = u' \mid L_n^0 = l, U_n^0 = u) = \nu(L_{n+1} = l', U_{n+1} = u' \mid L_n = l, U_n = u).$$

So though L_{n+1} and U_{n+1} are not independent, they can be coupled to independent random walks.

More precisely, let $C(l,u) = \{(l',u') : \max(l,l') \not< \min(u,u')\}$ (including $(\infty, -\infty)$). Then we can describe the evolution of the Markov chain (L_{n+1}, U_{n+1}) as follows: Starting from $(L_n, U_n) = (l, u)$ with $l < u$, propose a move to some (l', u') chosen with probabilities based on the independent random walks (L_{n+1}^0, U_{n+1}^0). If $(l', u') \notin C(l, u)$ the move is accepted. Otherwise the move is rejected, and replaced by a move to some point of $C(l, u)$ chosen according to the required law.

An elementary calculation shows that

$$\mathbb{E}[\Delta U_n^0] = \frac{p^3 - p^2 + 2p - 1}{p(1-p)}.$$

So assume that $p^3 - p^2 + 2p - 1 \leq 0$. Then by symmetry, $\mathbb{E}[\Delta(U_n^0 - L_n^0)] = 2\mathbb{E}[\Delta U_n] \leq 0$. Since $U_n^0 - L_n^0$ is a random walk, it therefore eventually enters $(-\infty, 0]$. In other words, ν-a.s. there will eventually be a rejected move.

Another elementary calculation shows that for $l < u$,

$$\begin{aligned} \nu(\text{ reject the next move } \mid L_n = l, U_n = u) &= [1 + (u-l)p](1-p)^{u-l} \\ \nu(L_{n+1} = U_{n+1} \mid L_n = l, U_n = u) &= [u - l + 1 - 2p]p(1-p)^{u-l}. \end{aligned}$$

Given $L_{n+1} = U_{n+1}$ there is probability $1-p$ that L_{n+2} and U_{n+2} will cross. We conclude that there is an $\epsilon > 0$ (depending on p but not u or l) such that every time a move is rejected, there is probability at least ϵ of a crossing within the next 2 steps of the chain. Since rejections keep recurring, it follows that ν-a.s., M_n^0 is eventually empty, and therefore so is M_n. In other words, $|\mathcal{C}_o| < \infty$.

The estimate on $p_c^{\underset{\downarrow}{\leftrightarrow}}$ now follows by computing the unique root of $p^3 - p^2 + 2p - 1$. $\square$

As remarked earlier, for the site percolation model of Example 1.2, the site percolation cluster of o is just $\mathcal{M}_o$. Moreover, $\mathcal{B}_o = \mathcal{M}_o$ provided $\mathcal{G}_o = \overset{\updownarrow}{\longleftrightarrow}$, and points in $\mathcal{C}_o$ are either in $\mathcal{M}_o$ or are neighbours of such points. These statements all follow because in this model, any connected path of $\overset{\updownarrow}{\longleftrightarrow}$ sites is necessarily connected in both directions. The following result is a kind of generalisation of this idea.

Lemma 2.12. *Suppose that there exists $A \neq \emptyset$ such that $\mu(\{A\}) = p$ and $\mu(\emptyset) = 1 - p$. Then $\nu(|\mathcal{C}_o| = \infty) = p\nu(|\mathcal{B}_o| = \infty)$.*

Proof. First note that

$$\{|\mathcal{C}_o| = \infty\} = \{\mathcal{G}_o = A\} \cap \Big\{\exists x_1 \in \{o + v : v \in A\} : |\mathcal{C}_{x_1}| = \infty \text{ in } \tilde{\mathcal{G}}\Big\}, \tag{2.2}$$

where $\tilde{\mathcal{G}}$ is the environment obtained from $\mathcal{G}$ by replacing $\mathcal{G}_o$ by $\emptyset$. The reason is that if the vertices in A each connect to only finitely many points, then adding in connections through o is not going to produce an infinite cluster. The two events on the right of (2.2) are independent since the first depends only on $\mathcal{G}_o$, while the latter depends only on $\tilde{\mathcal{G}}$.

Given an environment $\mathcal{G}$, let $\mathcal{G}^*$ denote the environment that we obtain from $\mathcal{G}$ by flipping every arrow (so $\mathcal{G}^*_x = \{-v : v \in \mathcal{G}_x\}$). Let $\mathcal{G}^{**}$ be the environment we obtain from $\mathcal{G}^*$ by flipping every arrow and vertex (so $\mathcal{G}^{**}_x = \{-v : v \in \mathcal{G}^*_{-x}\} = \mathcal{G}_{-x}$). Then $\mathcal{G}$ and $\mathcal{G}^{**}$ have the same law.

Now suppose that the second event on the right of (2.2) occurs. Then there is an infinite self-avoiding path of sites $o = x_0, x_1, x_2, x_3 \dots$ such that for each $i \geq 0$, $\mathcal{G}_{x_{i+1}} = A$ and $x_{i+1} = x_i + v_i$ for some $v_i \in A$. Thus in $\mathcal{G}^*$ the infinite self-avoiding path of sites $\dots x_3, x_2, x_1, x_0 = o$ is such that for each $i \geq 0$, $\mathcal{G}^*_{x_{i+1}} = -A$ and $x_i = x_{i+1} - v_i$ for some $-v_i \in -A$, and it follows that the path $\dots x_3, x_2, x_1, x_0 = o$ defines an infinite connected cluster to the origin in $\mathcal{G}^*$. Moreover, in $\mathcal{G}^{**}$, the path $\dots - x_3, -x_2, -x_1, -x_0 = o$ is such that for each $i \geq 0$, $\mathcal{G}^{**}_{-x_{i+1}} = A$ and $-x_i = -x_{i+1} + v_i$ for some $v_i \in A$, i.e. the path $\dots - x_3, -x_2, -x_1, -x_0 = o$ is an infinite connected cluster to the origin in $\mathcal{G}^{**}$. Similarly if there is an infinite connected cluster in $\mathcal{G}^{**}$ to the origin, then there is an infinite self-avoiding path of sites $\dots, x_3, x_2, x_1, x_0 = 0$ such that for each $i \geq 1$, $\mathcal{G}^{**}_{x_i} = A$ and $x_{i-1} = x_i + v_i$ for some $v_i \in A$. Reversing the procedure above, we construct an infinite connected path in $\mathcal{G}$ from a vertex $-x_1 = o + v_i$ for some $v_i \in A$. Thus we have shown that

$$\Big\{\exists x_1 \in \{o + v : v \in A\} : |\mathcal{C}_{x_1}| = \infty \text{ in } \tilde{\mathcal{G}}\Big\} = \Big\{|\mathcal{B}_o| = \infty \text{ in } \mathcal{G}^{**}\Big\}.$$

It follows that

$$\begin{aligned}\nu(|\mathcal{C}_o| = \infty\}) =& \nu(\{\mathcal{G}_o = A\})\nu\Big(\exists x_1 \in \{o + v : v \in A\} : |\mathcal{C}_{x_1}| = \infty \text{ in } \tilde{\mathcal{G}}\Big)\\ =& p\nu\Big(|\mathcal{B}_o| = \infty \text{ in } \mathcal{G}^{**}\Big) = p\nu\Big(|\mathcal{B}_o| = \infty \text{ in } \mathcal{G}\Big) = p\nu\Big(|\mathcal{B}_o| = \infty\Big).\end{aligned}$$

□

Like many results in this paper, the same result holds true for more general lattices. We will use this fact in the next section.

3 The set of points $\mathcal{B}_y$ from which y can be reached

There are cases in which points can only ever be reached from finitely many locations. This is known in the case of coalescing random walks (see [21]). Because we use the result repeatedly, we give the proof here.

Lemma 3.1. *Consider the model (↑→) of Example 1.5. Then for every y, $|\mathcal{B}_y| < \infty$, ν-almost surely.*

Proof. : Without loss of generality, $y = o$. Let $L_n = \{x : x^{[1]} + x^{[2]} = -n\}$. Set $X_n = \#\{x \in L_n : x \to o\}$, and let $\mathcal{F}_n$ reveal the environment on or above L_n. Then X_n is a positive martingale. In fact,

$$\begin{aligned}\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] &= \sum_{x \in L_{n+1}} \Big(p 1_{\mathcal{B}_o}((x^{[1]}, x^{[2]} + 1)) + (1-p) 1_{\mathcal{B}_o}((x^{[1]} + 1, x^{[2]})) \Big) \\ &= \sum_{y \in L_n \cap \mathcal{B}_o} \Big(p + (1-p) \Big) = X_n.\end{aligned}$$

Thus X_n converges as $n \to \infty$, and the only possible limit is 0. □

The following result is proved as in Lemma 2.4.

Lemma 3.2. *If* $\nu(|\mathcal{B}_o| = \infty) > 0$ *then* $\nu(\exists x : |\mathcal{B}_x| = \infty) = 1$.

We now turn to a class of results, giving environments under which $\mathcal{B}_o$ is infinite with positive probability. We start with a trivial criterion:

Lemma 3.3. *If there is an* e *such that* $\mu(\{A : e \in A\}) = 1$ *then all* $\mathcal{B}_x$ *are infinite almost surely.*

Proof. $\mathcal{B}_x \supset \{x - ne : n \geq 0\}$. □

This would be the case, for example, with

Example 3.4. $(\uparrow_{\rightarrow} \uparrow)$*:* $\mu(\{\uparrow, \rightarrow\}) = p$ *and* $\mu(\{\uparrow\}) = 1 - p$*, with* $0 < p < 1$.

As this example illustrates, when all $\mathcal{B}_x$ contain half-lines, it is also usually the case that the half-lines connect, and therefore that all $\mathcal{B}_x$ intersect.

More interesting are the cases we now turn to, in which $\mathcal{B}_x$ is infinite with probability between 0 and 1.

Example 3.5. $(\leftrightarrow\uparrow)$*:* $\mu(\{\leftarrow, \rightarrow\}) = p$ *and* $\mu(\{\uparrow\}) = 1 - p$*,* $0 < p < 1$.

Proposition 3.6. *Consider model* $(\leftrightarrow\uparrow)$ *of Example 3.5. Then*

(a) $1 > \nu(|\mathcal{B}_o| = \infty) > 0$*;*

(b) $\nu(\exists x : |\mathcal{B}_x| = \infty) = 1$*;*

(c) Almost surely, on the event that $\mathcal{B}_o$ *is infinite, there exists a semi-infinite path*

$$\ldots x_{-3} x_{-2} x_{-1} x_0$$

such that $x_0 = o$*,* $x_{-(n-1)} - x_{-n} \in \mathcal{G}_{x_{-n}}$ *for each* n*, and* $x_{-n}^{[2]} \to -\infty$ *(monotonically).*

(d) ν*-a.s., if* $|\mathcal{B}_x| = |\mathcal{B}_y| = \infty$ *then* $|\mathcal{B}_x \cap \mathcal{B}_y| = \infty$.

Proof. Let $\mathcal{L}_n = \mathbb{Z} \times \{-n\}$ and define L_n and U_n to be the infimum and supremum of the projection of $\mathcal{L}_n \cap \mathcal{B}_o$ on the 1st coordinate axis. Of course, if this set is empty then $L_n = \infty$ and $U_n = -\infty$. We claim that for each $n \geq 0$,

$$\mathcal{L}_n \cap \mathcal{B}_o = \Big([L_n, U_n] \cap \mathbb{Z}\Big) \times \{-n\}.$$

The claim is established by induction, with the case $n = 0$ being trivially true since there are no downward arrows.

Assume this statement for n, and suppose there is at least one $z \in [L_n, U_n]$ such that $\mathcal{G}_{(z,-(n+1))} = \{\uparrow\}$. Then $(z, -(n+1))$ connects to o as well, as does any $(w, -(n+1))$ which connects to $(z, -(n+1))$ by a sequence of $\to$ or $\leftarrow$. Thus $(w, -(n+1))$ connects to o whenever $L_n \leq w \leq U_n$, either directly or via such a z. This is also the case for any $w = (w_1, -(n+1))$ such that for every $k \in [w_1, L_n) \cap \mathbb{Z}$, the environment at $(k, -(n+1))$ is $\leftrightarrow$, but not other vertices to the left of $(L_n, -(n+1))$. Similarly for w's to the right of U_n at level $-(n+1)$. In other words, $L_{n+1} \leq L_n$ and $U_n \leq U_{n+1}$, and the set of $(w, -(n+1))$ connecting to o forms an interval.

On the other hand, if there is no $z \in [L_n, U_n]$ such that $\mathcal{G}_{(z,-(n+1))} = \{\uparrow\}$, then no vertex with 2nd coordinate $-(n+1)$ connects to o at all. Therefore for each n, either this interval expands ($L_{n+1} \leq L_n$ and $U_n \leq U_{n+1}$) or it disappears altogether ($= \emptyset$).

Consider the number of integers $D_n = (U_n - L_n + 1)_+$ in the interval $[L_n, U_n]$. It is easily checked that D_n is a Markov chain, that transitions $k \mapsto 0$ have probability $\alpha(k) = p^k$, that transitions $k \mapsto k$ have probability $(1-p)^2(1-p^k)$, and therefore that all other transitions combined have probability

$$\beta(k) = 1 - p^k - (1-p)^2(1-p^k) = (1-p^k)p(2-p) \geq (1-p)p(2-p) = c > 0.$$

Set $T_0 = 0$, and let $T_{k+1} = \min\{n > T_k : D_n \neq D_{T_k}\}$ be the times D_n changes values. Clearly $D_0 \geq 1$, and by induction, D_{T_k} is either $\geq k$ or it $= 0$. Therefore

$$\nu(D_{T_{k+1}} > D_{T_k} \mid \mathcal{F}_{T_k}) = \frac{\beta(D_{T_k})}{\alpha(D_{T_k}) + \beta(D_{T_k})} = 1 - \frac{\alpha(D_{T_k})}{\alpha(D_{T_k}) + \beta(D_{T_k})} \geq 1 - \frac{p^k}{c}.$$

Choose κ so large that $p^\kappa < c/2$, and γ such that $e^{-\gamma t} < 1 - t$ for $0 \leq t \leq 1/2$. Then the above expression is $\geq e^{-p^k \gamma / c}$ for $k \geq \kappa$. So by the strong Markov property, and convergence of $\sum p^k$,

$$\nu(D_n > 0 \quad \forall n) \geq \nu(D_{T_\kappa} > 0) \prod_{j=\kappa}^{\infty} e^{-p^k \gamma / c} > 0.$$

Thus in fact $D_n > 0$ for every n, with positive probability. Whenever all $D_n > 0$, it follows that $D_n \to \infty$ and $\mathcal{B}_o$ is infinite. In other words, we've proved that $\nu(|\mathcal{B}_o| = \infty) > 0$ (and (b) follows by Lemma 3.2). To see that it is < 1 as well, just observe that the configuration $\mathcal{G}_{(-1,0)} = \uparrow = \mathcal{G}_{(1,0)}$, $\mathcal{G}_{(0,-1)} = \leftrightarrow$ establishes that $\nu(\mathcal{B}_o = \{o\}) > 0$.

For future reference, notice that it follows from our proof that when $\mathcal{B}_o$ is infinite, it is almost surely also the case that $L_n \downarrow -\infty$ and $U_n \uparrow \infty$.

To obtain a semi-infinite path through $\mathcal{B}_o$, observe that we have at least one finite path from $(U_n, -n)$ to o, for each n. These can in fact be chosen to form a monotone sequence of paths, in

Figure 4: Parts of realisations of the set $\mathcal{B}_o$ for the model with $\mu(\{\leftarrow,\rightarrow\}) = p = 1-\mu(\{\uparrow\})$ for two values of p ($p = .7$ and $p = .3$).

the sense that if two such paths ever meet, we make them coalesce. It follows that the paths so chosen converge as $n \to \infty$. The limit is the desired semi-infinite path.

Finally, the fact that $\mathcal{B}_x \cap \mathcal{B}_y$ is infinite, whenever $\mathcal{B}_x$ and $\mathcal{B}_y$ are follows immediately from the monotonicity of L_n and U_n, and the fact that $D_n \to \infty$. $\square$

See Figure 4 for realisations of $\mathcal{B}_o$, for different values of p.

We now will establish the same type of result, for the following model.

Example 3.7. $(\uparrow\!\!\!\lrcorner\!\!\rightarrow\ \leftarrow)$: *$\mu(\{\uparrow,\rightarrow\}) = p$ and $\mu(\{\leftarrow\}) = 1-p$, $0 < p < 1$.*

Between them, Propositions 3.6 and 3.8 will allow us to decide whether $\nu(|\mathcal{B}_o| = \infty) > 0$ or not, for many 2-valued 2-dimensional models. See Table 2 for more details.

Proposition 3.8. *Consider model $(\uparrow\!\!\!\lrcorner\!\!\rightarrow\ \leftarrow)$ of Example 3.7. Then*

(a) $1 > \nu(|\mathcal{B}_o| = \infty) > 0$;

(b) $\nu(\exists x : |\mathcal{B}_x| = \infty) = 1$;

(c) Almost surely, on the event that $\mathcal{B}_o$ is infinite, there exists a semi-infinite path

$$\ldots x_{-3}x_{-2}x_{-1}x_0$$

such that $x_0 = o$, $x_{-(n-1)} - x_{-n} \in \mathcal{G}_{x_{-n}}$, and $x_{-n}^{[2]} \to -\infty$ (monotonically).

(d) ν-a.s., if $|\mathcal{B}_x| = |\mathcal{B}_y| = \infty$ then $|\mathcal{B}_x \cap \mathcal{B}_y| = \infty$.

Proof. For $n \geq 0$, let $\mathcal{L}_n$, L_n and U_n be defined as in the proof of Proposition 3.6. Let T be the first n (if any) such that $\mathcal{L}_n \cap \mathcal{B}_o = \emptyset$. As in Proposition 3.6 we will show by induction that

$$\mathcal{L}_n \cap \mathcal{B}_o = \Big([L_n, U_n] \cap \mathbb{Z}\Big) \times \{-n\}.$$

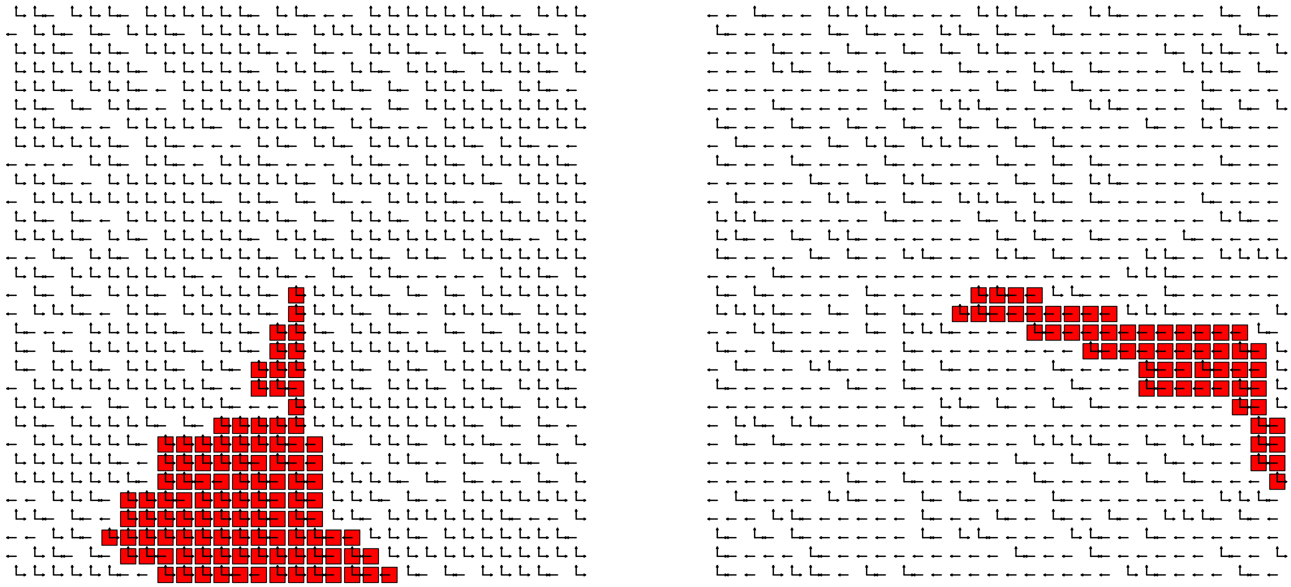

Figure 5: Parts of realisations of the set $\mathcal{B}_o$ for the model with $\mu(\{\uparrow,\rightarrow\}) = p = 1 - \mu(\{\leftarrow\})$ for two values of p ($p = .7$ and $p = .3$).

In other words, that the points in $\mathcal{B}_o \cap \mathcal{L}_n$ form a contiguous block, for $n < T$. See Figure 5.

So assume the above statement for n, and consider $\mathcal{B}_o \cap \mathcal{L}_{n+1}$. We will examine several cases separately. First, suppose $l \le L_n \le U_n \le u$. Then $L_{n+1} = l$ and $U_{n+1} = u$ if the following conditions hold:

$$\begin{aligned}
\mathcal{G}_{(l-1,-(n+1))} &= \leftarrow \\
\mathcal{G}_{(j,-(n+1))} &= \uparrow\!\!\rightarrow \qquad \text{for } l \le j \le L_n \\
\mathcal{G}_{(j,-(n+1))} &= \leftarrow \qquad \text{for } U_n < j \le u \\
\mathcal{G}_{(u+1,-(n+1))} &= \uparrow\!\!\rightarrow.
\end{aligned}$$

To see this, observe that $x = (j, -(n+1)) \notin \mathcal{B}_o$ for $j < l$, since tracing a path from x can only reach $\mathcal{L}_n$ strictly to the left of L_n, and hence outside $\mathcal{B}_o$. Likewise $x = (j, -(n+1)) \notin \mathcal{B}_o$ for $j > u$. For $l \le j \le L_n$ we can step to the right till we reach $(L_n, -(n+1))$ and then step up to $(L_n, -n)$. Thus $(j, -(n+1)) \in \mathcal{B}_o$. For $L_n \le j \le u$ consider a path that steps either left or up. The first step up will be at a point $(j', -(n+1))$ with $L_n \le j' \le U_n$, so by induction the point $(j', -n)$ we reach will lie in $\mathcal{B}_o$. Thus $\mathcal{B}_o \cap \mathcal{L}_{n+1}$ forms a contiguous block with this scenario.

Now suppose that $L_n < l \le U_n \le u$. Then the same argument shows that $L_{n+1} = l$ and $U_{n+1} = u$, and $(j, -(n+1)) \in \mathcal{B}_o$ for $l \le j \le u$, provided the following conditions hold:

$$\begin{aligned}
\mathcal{G}_{(j,-(n+1))} &= \leftarrow \qquad \text{for } L_n \le j < l \\
\mathcal{G}_{(l,-(n+1))} &= \uparrow\!\!\rightarrow \\
\mathcal{G}_{(j,-(n+1))} &= \leftarrow \qquad \text{for } U_n < j \le u \\
\mathcal{G}_{(u+1,-(n+1))} &= \uparrow\!\!\rightarrow.
\end{aligned}$$

Between them, the above scenarios cover all cases in which there is a $L_n \le j \le U_n$ with $\mathcal{G}_{(j,-(n+1))} = \llcorner$. So the only remaining possibility is that

$$\mathcal{G}_{(j,-(n+1))} = \leftarrow \qquad \text{for } L_n \le j \le U_n,$$

in which case $\mathcal{B}_o \cap \mathcal{L}_{n+1} = \emptyset$, so $L_{n+1} = +\infty$, $U_{n+1} = -\infty$. Thus we have inductively proved the above claim.

Let $l' \le u'$. We can now also read off $\nu(L_{n+1} = l, U_{n+1} = u \mid L_n = l', U_n = u')$, getting

$$\begin{aligned} (1-p)p^{l'-l+1}(1-p)^{u-u'}p = p^{l'-l+2}(1-p)^{u-u'+1}, &\quad \text{if } l \le l' \le u' \le u, \\ (1-p)^{l-l'}p(1-p)^{u-u'}p = p^2(1-p)^{l+u-l'-u'}, &\quad \text{if } l' < l \le u' \le u, \\ (1-p)^{u'-l'+1} &\quad \text{if } l = +\infty,\ u = -\infty. \end{aligned}$$

We now couple these to a pair of independent random walks L_n^0 and U_n^0 that evolve as follows: If $U_n^0 = u'$ then $U_{n+1}^0 = u' + j$ with probability $p(1-p)^j$, for $j \ge 0$. If $L_n^0 = l'$ and $j \ge 1$ then $L_{n+1}^0 = l' + j$ with probability $p(1-p)^j$. While if $L_n^0 = l'$ and $j \ge 0$ then $L_{n+1}^0 = l' - j$ with probability $p^{j+1}(1-p)$. It follows that for $l' \le u'$ we have

$$\nu(L_{n+1} = l, U_{n+1} = u \mid L_n = l', U_n = u') = \nu(L_{n+1}^0 = l, U_{n+1}^0 = u \mid L_n^0 = l', U_n^0 = u'),$$

provided $l \le u' \le u$. Put another way, let T^0 be the first n such that $L_n^0 > U_{n-1}^0$. Then the process $(L_n, U_n)_{0 \le n < T}$ has the same law as the process $(L_n^0, U_n^0)_{0 \le n < T^0}$.

Now consider statement (a) of the proposition. $\mathcal{B}_o$ is infinite exactly when $T = \infty$, so we want to show that $\nu(T^0 = \infty) > 0$. But T^0 is the first time the random walk $D_n^0 = L_n^0 - U_{n-1}^0$ hits $[1, \infty)$. So this result boils down to showing that the random walk D_n^0 drifts to the left, or in other words, that

$$\mathbb{E}[\Delta L_n^0] < \mathbb{E}[\Delta U_n^0].$$

In fact,

$$\mathbb{E}[\Delta L_n^0] = \sum_{j \ge 1} j(1-p)^j p - \sum_{j \ge 0} j p^{j+1}(1-p) = \frac{1-p}{p} - \frac{p^2}{1-p}$$

and

$$\mathbb{E}[\Delta U_n^0] = \sum_{j \ge 0} j(1-p)^j p = \frac{1-p}{p}.$$

So by the above reasoning, $\nu(|\mathcal{B}_o| = \infty) > 0$.

The other statements follow as in the proof of Proposition 3.6, with the exception of (d). To obtain (d) we need to use the law of large numbers, and the comparison with L_n^0 and U_n^0. This shows that whenever $\mathcal{B}_o$ is infinite, in fact $U_n/n \to \mathbb{E}[\Delta U_n^0] = (1-p)/p$, and likewise $L_n/n \to (1-p)/p - p^2/(1-p)$. So suppose $\mathcal{B}_x$ and $\mathcal{B}_y$ are both infinite. Without loss of generality, we'll assume that $x^{[2]} = y^{[2]}$ and $x^{[1]} < y^{[1]}$. Let $L_n(y)$ (resp. $U_n(x)$) be the lower (resp. upper) process obtained from our construction, starting not from o but from x (resp. y). Since the asymptotic speed of $L_n(y)$ is less than the asymptotic speed of $U_n(x)$, eventually $U_n(x) > L_n(y)$, providing common elements to $\mathcal{B}_x$ and $\mathcal{B}_y$. $\square$

Corollary 3.9. $1 > \nu(|\mathcal{B}_o| = \infty) > 0$ *and* $\nu(\exists x : |\mathcal{B}_x| = \infty) = 1$ *in the following cases as well:*

(a) model $(\uparrow\!\rightarrow\ \leftarrow\!\downarrow)$ *of Example 1.8;*

(b) model $(\leftrightarrow\updownarrow)$ *of Example 1.9;*

(c) model $(\overleftrightarrow{\downarrow}\ \uparrow)$ *of Example 1.10;*

Proof. The last two networks contain that of Proposition 3.6, while the first and third contain rotations of that of Proposition 3.8. This shows that $\nu(|\mathcal{B}_o| = \infty) > 0$. In each case it is simple to find trapping configurations showing $\nu(\mathcal{B}_o = \{o\}) > 0$, implying that $\nu(|\mathcal{B}_o| = \infty) < 1$. ☐

For $C \subset \mathcal{E}$, let

$$\mathcal{A}_C = \{A \subset \mathcal{E} : A \cap C \neq \emptyset\}.$$

We'll also use a shorthand notation such as $\mathcal{A}_{\uparrow\!\rightarrow}$ for $\mathcal{A}_{\{\uparrow,\rightarrow\}}$.

Proposition 3.10. *Fix* $d = 2$ *and suppose that* $\mu(\mathcal{A}_{\leftarrow\!\uparrow}) = 1$, $\mu(\mathcal{A}_{\downarrow\!\rightarrow}) = 1$, $\mu(\mathcal{A}_{\leftarrow}) > 0$, $\mu(\mathcal{A}_{\rightarrow}) > 0$, $\mu(\mathcal{A}_{\uparrow}) > 0$, *and* $\mu(\mathcal{A}_{\downarrow}) > 0$. *Then the following hold:*

(a) The following ν*-a.s. exhaust the possibilities for* $\mathcal{B}_x$*:*

- *(i)* $\mathcal{B}_x$ *is finite;*
- *(ii)* $\mathcal{B}_x = \mathbb{Z}^2$*;*
- *(iii) There exists a decreasing* $W : \mathbb{Z} \to \mathbb{Z}$ *such that* $\mathcal{B}_x = \{y : y^{[2]} \leq W(y^{[1]})\}$*;*
- *(iv) There exists a decreasing* $W : \mathbb{Z} \to \mathbb{Z}$ *such that* $\mathcal{B}_x = \{y : y^{[2]} \geq W(y^{[1]})\}$*.*

(b) Only one of (ii), (iii), (iv) can have probability different from 0.
More precisely, if $\nu(\text{(ii) holds for } \mathcal{B}_x) > 0$ *then* $\nu(\text{(iii) or (iv) hold for } \mathcal{B}_y) = 0$ *for every* y*, and similarly for cases (iii) and (iv).*

(c) ν*-a.s., if* $|\mathcal{B}_x| = |\mathcal{B}_y| = \infty$ *then* $|\mathcal{B}_x \cap \mathcal{B}_y| = \infty$*.*

Proof. Without loss of generality, $x = o$. Suppose $y, z \notin \mathcal{B}_o$, with $y^{[1]} = z^{[1]}$ and $y^{[2]} < z^{[2]}$. Because $\mu(\mathcal{A}_{\downarrow\!\rightarrow}) = 1$, we may find SE paths from both y and z that are consistent with the environment, but only use $\rightarrow$ or $\downarrow$. We wish to apply Lemma 2.7 (or more accurately, a rotation of that result). So our first task is to show that these SE paths can be chosen to arise from a model $\tilde{\mu}$ similar to those in Example 1.5. In other words, a model with $\tilde{\mu}(\{\downarrow\}) = q$ and $\tilde{\mu}(\{\rightarrow\}) = 1 - q$ for some $0 < q < 1$. A partition of $\mathcal{A}_{\downarrow\!\rightarrow}$ into disjoint sets $\mathcal{E}_1 \subset \mathcal{A}_{\downarrow}$ and $\mathcal{E}_2 \subset \mathcal{A}_{\rightarrow}$ with each $\mu(\mathcal{E}_i) > 0$ will produces such a $\tilde{\mu}$. Since $\mu(\mathcal{A}_{\downarrow}) > 0$, $\mu(\mathcal{A}_{\rightarrow}) > 0$ and $\mu(\mathcal{A}_{\downarrow} \cup \mathcal{A}_{\rightarrow}) = \mu(\mathcal{A}_{\downarrow\!\rightarrow}) = 1$, we can always split the atoms of μ between two events to arrange this, unless μ is actually concentrated on a single atom belonging to $\mathcal{A}_{\downarrow} \cap \mathcal{A}_{\rightarrow}$. In that case we can obtain our model $\tilde{\mu}$ by randomization – take the SE path by choosing between $\downarrow$ and $\rightarrow$ independently at random, using the same q at each vertex.

We may now apply (a rotation of) Lemma 2.7 to see that the SE path from y crosses the SE path from z with probability 1. Since these paths lie within $\mathcal{C}_y$ and $\mathcal{C}_z$ respectively, and both $y, z \in \mathcal{B}_o^c$, it follows that both paths lie entirely in $\mathcal{B}_o^c$ as well. Following one from y to the intersection point, and then the other backwards in time to z produces a simple polygonal path from y to z, all of whose vertices belong to $\mathcal{B}_o^c$.

The same argument applies to the left of o, so we may also find intersecting NW paths from y and z that use only the moves $\uparrow$ and $\leftarrow$. Following one path from z to the intersection point, and then the other back to y produces a simple polygonal path which also lies entirely within $\mathcal{B}_o^c$. Concatenating the two paths gives us a ring in $\mathcal{B}_o^c$ whose vertices lead from y to z and then back to y.

Now suppose that $w \in \mathcal{B}_o$, with $w^{[1]} = y^{[1]} = z^{[1]}$ but $y^{[2]} < w^{[2]} < z^{[2]}$. There is by definition a path from w to o which is consistent with the environment and therefore lies entirely in $\mathcal{B}_o$. This path cannot cross the above ring, from which we conclude that o is itself enclosed by the ring. It follows that $\mathcal{B}_o$ is also enclosed by the ring, and hence that $\mathcal{B}_o$ is finite. That is, in this scenario, condition (i) holds.

To put this a different way, suppose that $\mathcal{B}_o$ is infinite. The argument above establishes that for every $n \in \mathbb{Z}$, the set of points $y \in \mathcal{B}_o^c$ such that $y^{[1]} = n$ forms a vertical interval $\{n\} \times (L_n, U_n)$. Case (ii) above corresponds to this interval being empty for every n. So suppose further that the interval is non-empty for some n. Then constructing SE and NW paths from that point (as above) shows that the interval is in fact non-empty for every n. Even better, running the SE path backwards in time and the NW path forwards in time gives a simple polygonal path within $\mathcal{B}_o^c$ that crosses every vertical line in $\mathbb{Z}^2$.

If o lies below this path, then we must have $U_n = +\infty$ for every n, as any path to o from above our path would have to cross the latter. We will show that case (iii) holds with $W(n) = L_n$. With this choice of W we first show that $W(n) > -\infty$ for each n. Since $\mathcal{B}_o$ is assumed to be infinite, we have $W(n) > -\infty$ for some n. Suppose $W(m) = -\infty$ for some $m > n$. Choosing the smallest such m, we have $W(m-1) > -\infty$, and that $(m-1, k) \in \mathcal{B}_o$ for every $k \leq W(m-1)$, but that $(m, k) \notin \mathcal{B}_o$ for any k. In particular, there is no $\leftarrow$ in any $\mathcal{G}_{(m,k)}$, $k \leq W(m-1)$, since if there were, following that move from (m, k) would lead into $\mathcal{B}_o$, from which we could then reach o. This contradicts the assumption that $\mu(\mathcal{A}_{\leftarrow}) > 0$, since the latter easily implies that

$$\nu(\exists j_0, k_0 \text{ such that } \mathcal{G}_{(j_0,k)} \notin \mathcal{A}_{\leftarrow} \text{ for every } k \leq k_0) = 0.$$

A similar argument, using that $\mu(\mathcal{A}_{\rightarrow}) > 0$, rules out $W(m) = -\infty$ for $m < n$. It follows that $W(m) > -\infty$ for every m. In other words, $W : \mathbb{Z} \to \mathbb{Z}$.

To see that W is decreasing, consider $\mathcal{G}_{(n,W(n)+1)}$. By definition, $\downarrow \notin \mathcal{G}_{(n,W(n)+1)}$. Since $\mu(\mathcal{A}_{\downarrow\rightarrow}) = 1$, we must have that $\rightarrow \in \mathcal{G}_{(n,W(n)+1)}$. Therefore $W(n+1) \leq W(n)$.

Finally, if o lies above the constructed path, then the same argument shows that $-\infty = L_n < U_n < +\infty$ for each n, with U_n decreasing, which puts us in case (iv) with $W(n) = U_n$. This establishes (a) of the Proposition.

For simplicity we refer to case (iii) [resp. case (iv)] by saying that $\mathcal{B}_o$ is *blocked above* [resp. *blocked below*]. We say that a function $w : \mathbb{Z} \to \mathbb{Z}$ is an *upper blocking function* for $\mathcal{G}$ if for every $n \in \mathbb{Z}$:

- $\downarrow \notin \mathcal{G}_{(n,w(n)+1)}$;
- if $w(n) < w(n+1)$ then $\rightarrow \notin \mathcal{G}_{(n,k)}$ for $w(n) < j \leq w(n+1)$;

- if $w(n) > w(n+1)$ then $\leftarrow \notin \mathcal{G}_{(n+1,k)}$ for $w(n) \geq j > w(n+1)$.

Our function $W(n)$ in case (iii) above is clearly an upper blocking function where $w(n) < w(n+1)$ never arises.

With the obvious modifications, there is a comparable notion of a *lower blocking function*. We say that x lies *below* w if $x^{[2]} \leq w(x^{[1]})$, and *strictly below* w if $x^{[2]} < w(x^{[1]})$. There is likewise a notion of x being *(strictly) above* w. We know that in case (iii) [resp. case (iv)] $w = W$ is an upper [resp. lower] blocking function. In fact, the converse statements are nearly true: If an upper blocking function $w(\cdot)$ exists, then for each x below w, either $\mathcal{B}_x$ is finite, or $\mathcal{B}_x$ is blocked above. That is, cases (ii) and (iv) are ruled out. The reason is that any path to x from a vertex y strictly above w, that is consistent with the environment, must cross w somewhere. But then the local environment at that crossing point is inconsistent with the upper blocking condition. Similarly, if a lower blocking function w exists, then for each x above w, either $\mathcal{B}_x$ is finite, or $\mathcal{B}_x$ is blocked below.

To prove (b), suppose that $\nu(\mathcal{B}_o$ is blocked above$) \geq \delta > 0$. So

$$\nu(\exists \text{ an upper blocking function above } o) \geq \delta.$$

Choose $n \geq 1$. We may find a $k \geq 0$ such that

$$\nu(\exists \text{ an upper blocking function } w \text{ above } o\text{, such that } w(j) \geq -k \text{ for all } |j| \leq n) \geq \delta/2.$$

By translation invariance of ν, it follows that

$$\nu(\exists \text{ an upper blocking function } w \text{ such that } w(j) \geq n \text{ for all } |j| \leq n) \geq \delta/2$$

(just translate $\mathcal{G}$ upward by $k+n$). These are decreasing events, so in fact

$$\nu(\forall n \geq 1, \exists \text{ an upper blocking function } w_n \text{ such that } w_n(j) \geq n \text{ for all } |j| \leq n) \geq \delta/2.$$

But the latter is a tail event, in the sense that for any finite set of vertices S, it only depends on the environments at points outside S. So by the zero-one law,

$$\nu(\forall n \geq 1, \exists \text{ an upper blocking function } w_n \text{ such that } w_n(j) \geq n \text{ for all } |j| \leq n) = 1.$$

We conclude that $\nu(\mathcal{B}_y$ is finite or blocked above$) = 1$ for every y.

Likewise, if $\nu(\mathcal{B}_o$ is blocked below$) > 0$, it follows that $\nu(\mathcal{B}_y$ is finite or blocked below$) = 1$ for every y.

Finally, if $\nu(\mathcal{B}_o = \mathbb{Z}^2) > 0$ then $\nu(\exists$ upper blocking function above $o) < 1$. By translation invariance and what we have just shown, it follows that $\nu(\exists$ upper blocking function above $y) = 0$ for every y. Thus $\nu(\mathcal{B}_y$ is blocked above$) = 0$. Likewise $\nu(\mathcal{B}_y$ is blocked below$) = 0$. So by (a), $\nu(\mathcal{B}_y$ is finite or $= \mathbb{Z}^2) = 1$ for every y. This proves (b).

Moreover, (c) follows immediately from (a) and (b). □

Corollary 3.11. *For the model* $(\updownarrow, \leftrightarrow)$ *of Example 1.9, with* $0 < p < 1$*, we have that* $\nu(\mathcal{B}_o = \mathbb{Z}^2 \mid |\mathcal{B}_o| = \infty) = 1$.

Proof. By Corollary 3.9 the conditional probability is well defined. By symmetry, the events (iii) and (iv) in Proposition 3.10 (a) have the same probability. Therefore by assertion (b) this probability is zero. The result then follows from part (a). □

In the following, note the differences with Proposition 3.10; we have fewer possible cases, but on the other hand the path $W(n)$ is not claimed to be decreasing. We will see examples later, when in fact they aren't.

Corollary 3.12. *Fix $d = 2$. Assume that $\mu(\mathcal{A}_{\uparrow\rightarrow}) = 1$, $\mu(\mathcal{A}_{\leftarrow\uparrow}) = 1$, $\mu(\mathcal{A}_{\leftarrow}) > 0$, $\mu(\mathcal{A}_{\rightarrow}) > 0$, and $\mu(\mathcal{A}_{\uparrow}) > 0$. Then the following hold:*

(a) The following ν-a.s. exhaust the possibilities for $\mathcal{B}_x$:

- *(i) $\mathcal{B}_x$ is finite;*
- *(ii) $\mathcal{B}_x = \mathbb{Z}^2$;*
- *(iii) There exists a $W : \mathbb{Z} \to \mathbb{Z}$ such that $\mathcal{B}_x = \{y : y^{[2]} \leq W(y^{[1]})\}$.*

(b) Only one of (ii), (iii) can have probability different from 0.
More precisely, if $\nu($(ii) holds for $\mathcal{B}_x) > 0$ then $\nu($(iii) holds for $\mathcal{B}_y) = 0$ for every y, and vice versa.

(c) ν-a.s., if $|\mathcal{B}_x| = |\mathcal{B}_y| = \infty$ then $|\mathcal{B}_x \cap \mathcal{B}_y| = \infty$.

Proof. To obtain conclusions (a)-(c), simply replicate the proof of Proposition 3.10, using NE paths in place of SE paths. Everything goes through without change, except the property that $W(n)$ is decreasing.

To see that the analogue of case (iv) of Proposition 3.10 will not occur, suppose $\mathcal{B}_x$ is infinite and blocked below. Choose $y \in \mathcal{B}_x^c$ with $y^{[1]} = x^{[1]}$ and $y^{[2]} < x^{[2]}$. Follow the NE path from y till it reaches some point z with $z^{[2]} > x^{[2]}$. Then follow the NW path from z till it reaches some point $\tilde{z}$ with $\tilde{z}^{[1]} = x^{[1]}$. By construction, $\tilde{z}^{[2]} > x^{[2]}$ so by (iv) we must have $\tilde{z} \in \mathcal{B}_x$. But $y \to z \to \tilde{z} \to x$ implies that $y \to x$, which contradicts the fact that $y \notin \mathcal{B}_x$. Thus (iv) is impossible in this setting. □

In some cases, our previous work immediately rules out some of the possibilities. Another trivial result of this type is:

Corollary 3.13. *In addition to the hypotheses of Corollary 3.12, assume that $\mu(\mathcal{A}_{\downarrow}) = 0$. Then for each x, ν-a.s. either $\mathcal{B}_x$ is finite or it is blocked above.*

Proof. There is no path from y to x consistent with the environment, if $y^{[2]} > x^{[2]}$, which eliminates case (ii). □

In the remainder of this section, we explore some consequences of the above results for several models of particular interest.

Recall that in the triangular lattice, vertices lie at the intersections of three families of parallel lines. Each vertex has 6 neighbours,. To construct oriented site percolation on this lattice, one specifies a unit vector u in the direction of one of these families of lines. Closed vertices connect to no neighbours. Open vertices x connect to neighbours y such that $(y-x)\cdot u > 0$. So connections are to 0 neighbours or to 3 neighbours. If each vertex is open with probability p, independently of each other vertex, then there is a critical value $p_c^{\nwarrow\uparrow\leftarrow}$ such that oriented percolation clusters are finite when $p < p_c^{\nwarrow\uparrow\leftarrow}$ and are infinite with positive probability when $p > p_c^{\nwarrow\uparrow\leftarrow}$. The estimated value is $p_c^{\nwarrow\uparrow\leftarrow} \approx 0.5956$ (see De Bell and Essam [4] or Jensen and Guttmann [12]). We have not found rigorous bounds in the literature, though the following inequalities

$$0.5 \le p_c^{\nwarrow\uparrow\leftarrow} \le 0.7491 \tag{3.1}$$

can be inferred from bounds on other models. To be precise, if we decrease the allowed bonds we get $p_c^{\nwarrow\uparrow\leftarrow} \le p_c^{\uparrow\rightarrow} \le 0.7491$ (the latter due to Balister et al [2]). Similarly, if p_c^{TSP} denotes the critical threshold for (un-oriented) triangular site percolation, then $p_c^{\nwarrow\uparrow\leftarrow} \ge p_c^{\mathrm{TSP}} = 1/2$ (see Hughes [11]). We will improve on the lower bound in Section 4.

Theorem 3.14. *Consider the model* $(\uparrow\rightarrow\ \leftarrow\downarrow)$ *of Example 1.8.*

(a) $0 < \nu(|\mathcal{B}_o| = \infty) < 1$.

(b) If $1 - p_c^{\nwarrow\uparrow\leftarrow} < p < p_c^{\nwarrow\uparrow\leftarrow}$ *and* $\mathcal{B}_o$ *is infinite, then* $\mathcal{B}_o = \mathbb{Z}^2$.

(c) If $p > p_c^{\nwarrow\uparrow\leftarrow}$ *and* $\mathcal{B}_o$ *is infinite, then it is blocked above. Moreover the blocking function* W *is decreasing.*

(d) If $p < 1 - p_c^{\nwarrow\uparrow\leftarrow}$ *and* $\mathcal{B}_o$ *is infinite, then it is blocked below. Moreover the blocking function* W *is decreasing.*

Proof. Statement (a) just reiterates part (a) of Corollary 3.9. Observe further that the hypotheses of Proposition 3.10 hold.

Now let $w(n)$ be a decreasing function, and consider under what circumstances it can be an upper blocking function. Let

$$w_{\le} = \{y : y^{[2]} \le w(y^{[1]})\}$$

denote the region under w. Vertices in $w_{\le}^c$ which border $w_{\le}$ lie above or to the right of $w_{\le}$, and can be enumerated naturally to form a sequence of vertices moving upwards and to the left. More precisely, the possible transitions in this sequence of vertices are as follows.

- Upwards, e.g. from (n,k) to $(n,k+1)$. This happens if $w(n) < k < w(n-1)$.
- Leftwards, e.g. from (n,k) to $(n-1,k)$. This happens if $w(n) = w(n-1) = k-1$.
- Diagonally to the NW, e.g. from (n,k) to $(n-1,k+1)$. This happens if $w(n) < k = w(n-1)$.

We recognize these as three of the six possible transitions in a triangular lattice. The three families of lines are horizontal, vertical, and diagonal with slope -1, though the set of lattice points is still $\mathbb{Z}^2$. In other words, the sequence of vertices we obtain are exactly the sequences that arise from oriented paths in this triangular lattice. For $w(n)$ to be an upper blocking function, it is necessary and sufficient that each vertex in this sequence have local environment $\uparrow\rightarrow$. Calling $\uparrow\rightarrow$ vertices "open" and $\leftarrow\downarrow$ vertices "closed", we have established the kind of duality relation that is familiar

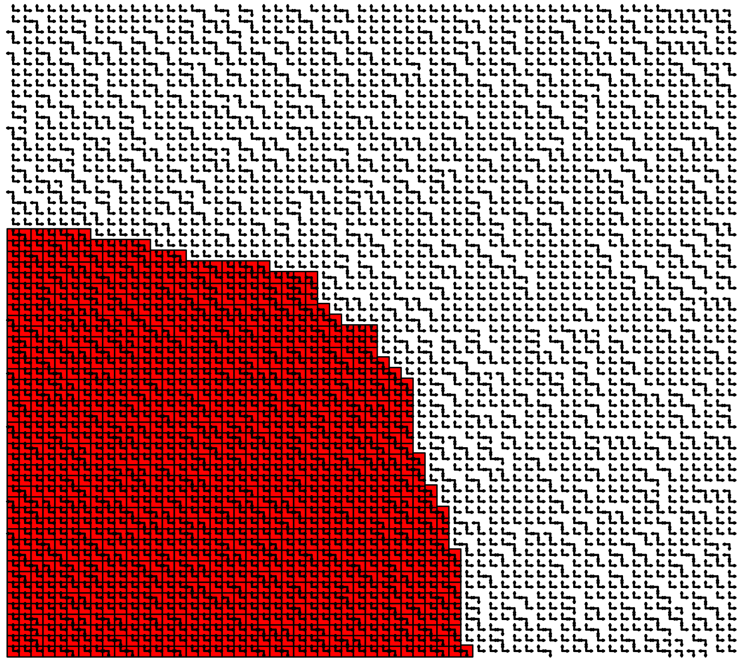

Figure 6: Part of a realisation of the set $\mathcal{B}_o$ for the model with $\mu(\{\uparrow, \rightarrow\}) = .8 = 1 - \mu(\{\leftarrow, \downarrow\})$.

from percolation: upper blocking functions for our random environment correspond precisely to clusters for *oriented triangular site percolation* (a model which we denote $\searrow\!\!\uparrow$).

Thus upper blocking functions exist if and only if there are doubly infinite oriented percolation clusters. In other words, if and only if there are points x such that $\mathcal{B}_x^{\searrow\!\uparrow}$ and $\mathcal{C}_x^{\searrow\!\uparrow}$ are both infinite (where the superscript indicates that we are referring to the $\searrow\!\!\uparrow$ model described above). We claim that this occurs when $p > p_c^{\searrow\!\uparrow}$, but not when $p < p_c^{\searrow\!\uparrow}$. To see this, observe that the argument of Lemma 2.12 applies equally well to the lattice $\searrow\!\!\uparrow$. If $p > p_c^{\searrow\!\uparrow}$ then $\nu(|\mathcal{C}_x^{\searrow\!\uparrow}| = \infty) > 0$ for each x, and so $\nu(|\mathcal{B}_x^{\searrow\!\uparrow}| = \infty) > 0$. It follows from the FKG inequality (see e.g. Grimmett [8, Theorem 2.2]), or simply because these two events depend on disjoint sets of vertices, that $\nu(|\mathcal{B}_x^{\searrow\!\uparrow}| = \infty = |\mathcal{C}_x^{\searrow\!\uparrow}|) > 0$. To complete the argument, check vertices to the NW and SE of the origin (according to some enumeration taking vertices close to the origin first). We either get that $|\mathcal{B}_o^{\searrow\!\uparrow}| = \infty = |\mathcal{C}_o^{\searrow\!\uparrow}|$, or after checking only k vertices for some finite k, we find that one of these sets is finite. In the latter case we may repeat this procedure from the vertex $(k+1, k+1)$ from which we see a completely new environment. Continuing in this way we almost surely eventually find a vertex x with $|\mathcal{B}_x^{\searrow\!\uparrow}| = \infty = |\mathcal{C}_x^{\searrow\!\uparrow}|$. This establishes part (c).

A similar argument (or just using symmetry) gives part (d), and (b) follows immediately from the above argument and (a) of Proposition 3.10. Note that (b) of Proposition 3.10 implies that $p_c^{\searrow\!\uparrow} \geq 1/2$, as we can't simultaneously have both upper and lower blocking functions. That is, we recover the lower inequality from (3.1). $\square$

Now consider the following *oriented next-nearest-neighbour square site percolation model* $\nearrow\!\!\downarrow$: vertices are $\mathbb{Z}^2$, edges are vertical, horizontal, or diagonal (with slope ± 1), so there are 8 neighbours to each vertex. There are 5 oriented edges from each vertex: up, down, to the left, and diagonally to the NW or SW. Denote the critical p for oriented percolation on the next-nearest-neighbour square lattice by $p_c^{\nearrow\!\downarrow}$. We have not found numerical estimates for $p_c^{\nearrow\!\downarrow}$ in the literature, and the best bounds available seem to be

$$0.3205 \le p_c^{\nwarrow\!\!\swarrow} \le 0.7491; \tag{3.2}$$

One obtains the upper bound from the inequalities $p_c^{\nwarrow\!\!\swarrow} \le p_c^{\swarrow\!\!\nearrow} \le p_c^{\hookrightarrow}$ and the bound of Balister et al [2] on the latter. The lower bound comes from the inequality $p_c^{\nwarrow\!\!\swarrow} \ge 1 - p_c^{\boxplus}$ (obtained by the duality between the square lattice and the next-nearest-neighbour square lattice – see Russo [20]), and the upper bound on $p_c^{\boxplus}$ of Wierman [23]. We will improve on the lower bound in Section 4.

Theorem 3.15. *Consider the model* ($\leftrightarrow\!\downarrow$ $\uparrow$) *of Example 1.10.*

(a) $0 < \nu(|\mathcal{B}_o| = \infty) < 1$.

(b) If $p > 1 - p_c^{\nwarrow\!\!\swarrow}$ *and* $\mathcal{B}_o$ *is infinite, then* $\mathcal{B}_o = \mathbb{Z}^2$.

(c) If $p < 1 - p_c^{\nwarrow\!\!\swarrow}$ *and* $\mathcal{B}_o$ *is infinite, then it is blocked above.*

Proof. Statement (a) just reiterates part (c) of Corollary 3.9. Next, observe that the hypotheses of Corollary 3.12 hold. Following the previous argument, we let $w : \mathbb{Z} \to \mathbb{Z}$ be any function (not necessarily monotone; see Figure 7), and consider when it can be an upper blocking function. Again, vertices in $w_\le^c$ which border $w_\le$ now can lie immediately above, to the right, or to the left of points in $w_\le$. Such vertices can again be enumerated to form a sequence, though this time there can be repetition. More precisely, the possible transitions in this sequence are as follows.

- Upwards, e.g. from (n,k) to $(n,k+1)$. This happens if $w(n) < k < w(n-1)$.
- Downwards, e.g. from (n,k) to $(n,k-1)$. This happens if $w(n+1) > k > w(n)$.
- Leftwards, e.g. from (n,k) to $(n-1,k)$. This happens if $w(n) = w(n-1) = k-1$.
- Diagonally to the NW, e.g. from (n,k) to $(n-1,k+1)$. This happens if $w(n) < k = w(n-1)$.
- Diagonally to the SW, e.g. from (n,k) to $(n-1,k-1)$. This happens if $w(n-1) < w(n) = k-1$.

We recognize these as the oriented transitions in the lattice $\nwarrow\!\!\swarrow$ described above. In other words, the sequences of vertices we obtain are exactly the sequences that arise from oriented paths in this next-nearest-neighbour square lattice. For $w(n)$ to be an upper blocking function, it is necessary and sufficient that each vertex in this sequence have local environment $\uparrow$, since at least one arrow of $\leftrightarrow\!\downarrow$ would violate the blocking condition. Calling $\uparrow$ vertices "open" and $\leftrightarrow\!\downarrow$ vertices "closed", we have established the same kind of duality relation as before: upper blocking functions for our random environment correspond precisely to doubly infinite oriented paths in percolation clusters for $\nwarrow\!\!\swarrow$. By the FKG inequality, $\nu(|\mathcal{B}_o^{\nwarrow\!\!\swarrow}| = \infty = |\mathcal{C}_o^{\nwarrow\!\!\swarrow}|) > 0$. To complete the proof of part (c) we need to show that $\nu(\exists x : |\mathcal{B}_x^{\nwarrow\!\!\swarrow}| = \infty = |\mathcal{C}_x^{\nwarrow\!\!\swarrow}|) = 1$. Let $A_{n,k}(x)$ be the event that for some $0 \le r \le n$, $\mathcal{G}_{x+y} = \uparrow$ for all $y \in \{(0,r),(1,r),\dots,(k,r)\}$ (in other words, that in the $\nwarrow\!\!\swarrow$ version of the environment, there is a horizontal block of "open" vertices). Then $\nu(\cup_{n=1}^\infty A_{n,k}(o)) = 1$, and hence

$$\nu\Big(|\mathcal{B}_o^{\nwarrow\!\!\swarrow}| = \infty = |\mathcal{C}_o^{\nwarrow\!\!\swarrow}| \cap (\cup_{n=1}^\infty A_{n,k}(o))\Big) = \delta > 0.$$

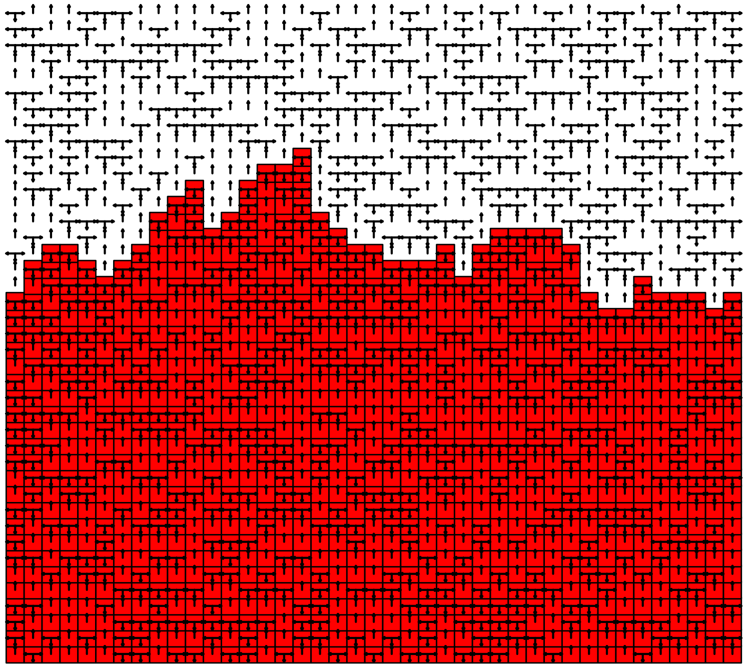

Figure 7: Part of a realisation of the set $\mathcal{B}_o$ for the model with $\mu(\{\downarrow, \leftarrow, \rightarrow\}) = .5 = \mu(\uparrow)$.

It follows that for every k there exists $n_0(k)$ such that $\nu\Big(|\mathcal{B}_o^{\nwarrow\!\searrow}| = \infty = |\mathcal{C}_o^{\nwarrow\!\searrow}| \cap A_{n_0(k),k}(o))\Big) > \frac{\delta}{2}$. Now as in Theorem 3.14 check vertices close to the origin to see if $|\mathcal{B}_o^{\nwarrow\!\searrow}| = \infty = |\mathcal{C}_o^{\nwarrow\!\searrow}|$. If either of these sets turns out to be finite (after checking only k_1 vertices for some finite k_1) then conduct a second search from the point x_1, where $x_1^{[1]}$ is 1 less than the first coordinate of a leftmost vertex already checked and $x_1^{[2]}$ is the second coordinate of a southmost vertex already checked minus $n_o(k_1+1)+1$. The new search seeks to determine if $|\mathcal{B}_{x_1}^{\nwarrow\!\searrow}| = \infty = |\mathcal{C}_{x_1}^{\nwarrow\!\searrow}| \cap A_{n_0(k_1+1),k_1+1}(x_1))$ occurs. The point is that, if there is a horizontal block of k_1+1 "open" vertices, then this search need not examine any vertex looked at in the first search. Now iterate. Each time we fail to find our vertex, we have only visited a finite number of vertices, and we proceed to find an x with our desired property by only looking at unexplored vertices (chosen depending on the finite set of vertices visited so far) with a chance of at least $\frac{\delta}{2}$ of being successful on each pass. Eventually we succeed. Thus upper blocking functions exist if and only if there are doubly infinite oriented percolation clusters. In other words, there are upper blocking functions when $(1-p) > p_c^{\nwarrow\!\searrow}$, as required.

Finally, if $(1-p) < p_c^{\nwarrow\!\searrow}$ then upper blocking functions do not exist, and the only alternative remaining from Corollary 3.12 is that the infinite $\mathcal{B}$ all equal $\mathbb{Z}^2$. This shows (b). □

Of the 2-valued models in $d = 2$, which ones exhibit the kind of phase transitions of $\mathcal{B}_x$ that we have been examining? In other words, when does $\mathcal{B}_x = \mathbb{Z}^2$ happen for some p but not others? We have seen two such models already: ($\uparrow\!\rightarrow$, $\leftarrow\!\downarrow$) and ($\leftarrow\!\downarrow\!\rightarrow$, $\uparrow$). It turns out that there are precisely three more (modulo rotations and reflections) – see Table 2. The following example describes what happens for each of them. We don't give a proof, since the arguments are simple modifications of ones given already. In each case, p is the probability of the first listed local configuration. Note that in these models it is not possible for $\mathcal{B}_o$ to be finite, since Lemma 3.3 shows that each $\mathcal{B}_o$

contains a half line. Thus the alternatives are being $\mathbb{Z}^2$ or being blocked.

Example 3.16. *For $d = 2$, the following models have phase transitions as shown.*

(a) (⭥↔ ⮡): $\nu(\mathcal{B}_o = \mathbb{Z}^2) = 1$ *when* $1 > p > 1 - p_c^{\nwarrow\searrow}$ *;* $\mathcal{B}_o$ *is blocked above when* $p < 1 - p_c^{\nwarrow\searrow}$ *.*

(b) (↔↕ ⮡): $\nu(\mathcal{B}_o = \mathbb{Z}^2) = 1$ *when* $1 \geq p > 1 - p_c^{\nwarrow\searrow}$ *;* $\mathcal{B}_o$ *is blocked above when* $p < 1 - p_c^{\nwarrow\searrow}$ *.*

(c) (↔↕ ↑): $\nu(\mathcal{B}_o = \mathbb{Z}^2) = 1$ *when* $1 \geq p > 1 - p_c^{\nwarrow\searrow\updownarrow}$*;* $\mathcal{B}_o$ *is blocked above when* $p < 1 - p_c^{\nwarrow\searrow\updownarrow}$*.*

Note that one can prove duality-type results analogous to those in this section, for the sets $\mathcal{C}_x$, as well as results about the asymptotic shape of $\mathcal{B}_x$ or $\mathcal{C}_x$ when these are blocked above or below. We do not include these arguments here since they are not required for the main results of this paper. We hope to include them in a subsequent paper.

4 The mutually connected clusters $\mathcal{M}_x = \mathcal{C}_x \cap \mathcal{B}_x$

In this section we examine the sets of points that are mutually connected. In many cases, the following immediately shows that $\mathcal{M}_x$ is finite.

Lemma 4.1. *Suppose there is some e such that $\nu(\{A : e \in A\}) > 0$ but $\mu(\{A : -e \in A\}) = 0$. Then $\mathcal{M}_x \subset \{y : (y - x) \cdot e = 0\}$ a.s., for every x.*

Proof. Once a path consistent with the environment leaves this set, it can never return. □

This shows that the $\mathcal{M}$ are finite ν-a.s. in such cases as model (↔↑) of Example 3.4 or model (⮡ ⮢) of Example 1.1.

On the other hand, the following is a non-trivial condition on μ which guarantees that $\mathbb{E}[|\mathcal{M}_o|] < \infty$, and hence that $|\mathcal{M}_o| < \infty$, ν-almost surely.

Theorem 4.2. *For each $d \geq 2$ there exists ϵ_d such that the following holds: If there exists an orthogonal set V of unit vectors such that $\mu(\{A : \emptyset \neq A \subset V\}) > 1 - \epsilon_d$, then $\mathbb{E}[|\mathcal{M}_o|] < \infty$.*

Proof. Without loss of generality, assume that $V = \{e_1, \dots, e_k\}$ for some $k \leq d$, and let $v = \sum_{u \in V} u$. Let L^+ (resp. L^-) be the set of points $x \in \mathbb{Z}^d$ such that $x \cdot v \geq 0$ (resp. ≤ 0). Let $L_N^+ = L^+ \cap \{x : \|x\|_1 = N\}$ and similarly for L_N^-. Note that $\{x : \|x\|_1 < r\} \cup \bigcup_{N=r}^{\infty}(L_N^+ \cup L_N^-) = \mathbb{Z}^d$.

We claim that for ϵ sufficiently small, there exist $c > 0$ and $\eta \in (0, 1)$ such that for N sufficiently large,

$$\nu(\mathcal{C}_o \cap L_N^- \neq \emptyset) < c\eta^N. \tag{4.1}$$

Assume for the moment that this is true. For $x \in L_N^-$, $\nu(x \in \mathcal{C}_o) \leq \nu(\mathcal{C}_o \cap L_N^- \neq \emptyset)$. Likewise if $x \in L_N^+$ then by translation invariance, $\nu(o \in \mathcal{C}_x) = \nu(-x \in \mathcal{C}_o) \leq \nu(\mathcal{C}_o \cap L_N^- \neq \emptyset)$. For $r \gg 1$, this implies that

$$\mathbb{E}[|\mathcal{M}_o|] =\mathbb{E}\left[\sum_{x\in\mathbb{Z}^d} I_{\{x\in\mathcal{C}_o\}} I_{\{o\in\mathcal{C}_x\}}\right] \le \sum_{x\in\mathbb{Z}^d} \nu(x\in\mathcal{C}_o)^{\frac{1}{2}}\nu(o\in\mathcal{C}_x)^{\frac{1}{2}} \tag{4.2}$$

$$\le \sum_{x:\|x\|_1<r} 1 + \sum_{N=r}^{\infty}\left(\sum_{x\in L_N^-}\nu(x\in\mathcal{C}_o)^{\frac{1}{2}} + \sum_{x\in L_N^+}\nu(o\in\mathcal{C}_x)^{\frac{1}{2}}\right) \tag{4.3}$$

$$\le C_d r^d + c'\sum_{N=r}^{\infty}\left(\sum_{x\in L_N^-}\eta^{\frac{N}{2}} + \sum_{x\in L_N^+}\eta^{\frac{N}{2}}\right) \le C_d r^d + \sum_{N=r}^{\infty} C_d' N^{d-1}\eta^{\frac{N}{2}} < \infty. \tag{4.4}$$

Hence it only remains to verify the claim of (4.1).

If $\mathcal{C}_o \cap L_N^- \neq \emptyset$ then there exists x such that $\sum_{i=1}^k x_i \le 0$ and $\|x\|_1 = N$, and there is an admissible (recall Definition 1.7) self-avoiding path from o to x. Moreover any finite path connecting o to x must consist of at least N steps, with at least half of the steps of the path being taken in the directions taken from $\{e_1,\dots,e_k\}^c$ (if more than half of the steps of a finite path are taken from $\{e_1,\dots,e_k\}$, then the endpoint y of the path must have $\sum_{i=1}^k y_i > 0$).

Let c_N denote the number of self-avoiding walks of length N starting from the origin, and let μ_d denote the self-avoiding walk connective constant in d dimensions, defined as

$$\mu_d := \lim_{N\to\infty} c_N^{\frac{1}{N}}.$$

The probability that at least one of these is an admissible path, at least half of whose steps are in the directions $\{e_1,\dots,e_k\}^c$ is therefore at most c_N times the maximum (over paths) probability that a particular such path is admissible. Any such path has at least $\lfloor N/2\rfloor$ steps in $\{e_1,\dots,e_k\}^c$, each of which is admissible with probability at most ϵ. It follows that the probability that there is an admissible self-avoiding path of length N starting from the origin, at least half of whose steps are in the directions $\{e_{k+1},\dots,e_d\}$ is at most

$$c_N\epsilon^{\lfloor\frac{N}{2}\rfloor} \le C\eta^N,$$

provided $\sqrt{\epsilon}\mu_d < \eta$. Accordingly, if $\epsilon < 1/\mu_d^2$, we can find such an $\eta < 1$, which completes the proof (with $\epsilon_d = 1/\mu_d^2$). □

Note that we can improve on ϵ_d with more information about μ. For example, suppose in addition to the hypotheses of the Lemma, we have a dichotomy

$$\mu(\{A : A\subset V \text{ or } A\subset V^c\}) = 1.$$

Then as long as $\epsilon < 1/2$, the bound on the relevant probability becomes

$$c_N\epsilon^{\lfloor\frac{N}{2}\rfloor}(1-\epsilon)^{N-\lfloor\frac{N}{2}\rfloor},$$

and solving the quadratic inequality, we find an $\eta < 1$ giving the conclusion of the theorem, provided

$$\epsilon < \epsilon_d^0 = \frac{1}{2}\left(1-\sqrt{1-\frac{4}{\mu_d^2}}\right).$$

d	rigorous	estimate
2	$2.6200 \le \mu_2 \le 2.6792$	2.6381585
3	$4.5721 \le \mu_3 \le 4.7114$	4.68404
4	$6.7429 \le \mu_4 \le 6.8040$	6.77404
5	$8.8285 \le \mu_5 \le 8.8602$	8.83854

Table 1: Rigorous lower and upper bounds and numerical estimates for the self-avoiding walk connective constant μ_d for $d = 2, 3, 4, 5$, see Finch [7, Table 5.2].

See Table 1 for values of μ_d for $d = 2, 3, 4, 5$. For $d = 2$, the rigorous upper bound on μ_2 gives values $\epsilon_2 = 0.13931$ and $\epsilon_2^0 = 0.16730$

This immediately implies the following.

Corollary 4.3. *All $\mathcal{M}_x$ are finite, in the following cases:*

(a) the (↿↳ ↰↴) model of Example 1.8, whenever $p > 1 - \epsilon_2^0 = 0.83270$ or $p < \epsilon_2^0 = 0.16730$;

(b) the (⊤↔↓ ↑) model of Example 1.10, whenever $p < \epsilon_2^0 = 0.16730$;

Corollary 4.4. $p_c^{⤡↕} \le p_c^{⤡} \le 1 - \epsilon_2^0 = 0.83270$

Proof. For the (↿↳ ↰↴) model we know $\mathcal{B}_x = \mathbb{Z}^2$ with positive probability if $1 - p_c^{⤡} < p < p_c^{⤡}$, in which case $\mathcal{M}_x = \mathcal{C}_x$ is infinite. Thus $p_c^{⤡} \le 1 - \epsilon_2^0$. We could get an upper bound on $p_c^{⤡↕}$ the same way, using the (⊤↔↓ ↑) model. But that is subsumed in the inequality $p_c^{⤡↕} \le p_c^{⤡}$, which follows by inclusion between the lattices. □

The above Corollaries serve to illustrate the general approach of the Theorem. But they do not in fact improve on the upper bound in (3.1). For further 2-dimensional examples, see Table 2. Generalizing part (a) of Corollary 4.3 to higher dimensions we have:

Example 4.5 (orthant model)**.** *$A = \{+e_i : i = 1, \dots, d\}$, $\mu(\{A\}) = p$, and $\mu(\{A^c\}) = 1 - p$. Then all $\mathcal{M}_x$ are finite whenever $p < \epsilon_d^0$ or $p > 1 - \epsilon_d^0$.*

We now turn to some models for which we can show that there are infinite mutually-connected clusters. As before, this is trivial in some cases:

Lemma 4.6. *Suppose there is an e such that $\mu(\{A : e \in A, -e \in A\}) = 1$. Then all $\mathcal{M}_x$ are infinite almost surely.*

Proof. $\mathcal{M}_x \supset \{x + ne : n \in \mathbb{Z}\}$. □

This is the case, for example, in the model (⊤↔↓ ↔) Sometimes we can go further:

Lemma 4.7. *Suppose that for every e, $\mu(\{A : e \in A\}) > 0$, and that for some $\tilde{e}$, $\mu(\{A : \tilde{e} \in A, -\tilde{e} \in A\}) = 1$. Then $\mathcal{M}_x = \mathbb{Z}^d$ for every x.*

Proof. Any line in the direction $\tilde{e}$ is contained in an $\mathcal{M}_x$, and connects to every neighbouring such line. □

A 2-dimensional example where Lemma 4.7 applies is the following.

Example 4.8. $(\overset{\leftrightarrow}{\downarrow}\,\underset{\leftrightarrow}{\uparrow})$: *$\mu(\{\rightarrow, \downarrow, \leftarrow\}) = p$ and $\mu(\{\rightarrow, \uparrow, \leftarrow\}) = 1 - p$, where $0 < p < 1$.*

In the more interesting cases, which we now turn to, there will be one infinite cluster $\mathcal{M}_x$, and an infinite number of finite clusters $\mathcal{M}_y$. We start with the following definition and observations:

Definition 4.9. *If there is an x such that $\mathcal{M}_x$ is infinite and all $\mathbb{Z}^d$-connected components of $\mathbb{Z}^d \setminus \mathcal{M}_x$ are finite, then we say that $\mathcal{G}$ has a* giant $\mathcal{M}$-component.

Lemma 4.10. *Assume (2.1). Assume that $\mathcal{G}$ has a giant $\mathcal{M}$-component $\mathcal{M}_x$. Then*

(a) $\mathcal{M}_x$ is the only infinite equivalence class for the communication relation.

(b) $\mathcal{C}_y \supset \mathcal{M}_x$ for every y, so all $\mathcal{C}_y$ intersect. Also $\mathcal{C}_y = \mathcal{M}_x$ for $y \in \mathcal{M}_x$.

(c) $\mathcal{B}_y = \mathbb{Z}^d$ for $y \in \mathcal{M}_x$, and $\mathcal{B}_y$ is finite otherwise.

(d) For each y, $\nu(|\mathcal{M}_y| = \infty) > 0$.

Proof. (a) is immediate, since each $\mathcal{M}_y$ is connected, and either coincides with $\mathcal{M}_x$ or is disjoint from it. In the latter case, it is therefore contained in a component of $\mathbb{Z}^d \setminus \mathcal{M}_x$, which is finite by hypothesis. By (2.1), all $\mathcal{C}_y$ are infinite, so leave finite components of $\mathbb{Z}^d \setminus \mathcal{M}_x$, and hence intersect $\mathcal{M}_x$. This implies (b), and likewise (c). Part (d) follows by translation invariance of ν. □

Corollary 4.11. *Assume the hypotheses of Proposition 3.10 or Corollary 3.12. There is a giant $\mathcal{M}$-component $\iff \nu(\mathcal{B}_o = \mathbb{Z}^2) > 0$.*

Proof. Note that the hypotheses of Proposition 3.10 or Corollary 3.12 imply (2.1). So the $\Rightarrow$ direction just reiterates (c) of Lemma 4.10. So we must prove $\Leftarrow$.

First, assume the hypotheses of Corollary 3.12. Suppose $\{\mathcal{B}_o = \mathbb{Z}^2\}$ has positive probability. Then ν-a.s. $\mathcal{B}_y$ is infinite for some y by (c), in which case $\mathcal{B}_y = \mathbb{Z}^2$ by (a) and (b). Therefore $\mathcal{M}_y = \mathcal{C}_y$ is infinite by (2.1). Likewise any other z with $\mathcal{M}_z$ infinite automatically has $\mathcal{B}_z = \mathbb{Z}^2$, and hence $\mathcal{M}_z = \mathcal{M}_y$ (as we can get $y \leftrightarrow z$). Thus there is only one infinite equivalence class for the communication relation, which we denote by $\mathcal{M}$. It remains to show that $\mathcal{M}$ is giant, that is, that connected components of $\mathbb{Z}^2 \setminus \mathcal{M}$ are finite.

For $k \in \mathbb{Z}$ let ℓ_k be the line $\{x^{[1]} = k\}$. We first show that $\mathcal{M} \cap \ell_k$ is a.s. non-empty. Suppose there is positive probability of $\mathcal{M} \cap \ell_k = \emptyset$. $\mathcal{M}$ is connected, so without loss of generality there is positive probability that $\mathcal{M}$ lies completely to the right of ℓ_k. Let

$$K = \sup\{k : \mathcal{M} \text{ lies completely to the right of } \ell_k\}.$$

What we've shown is that K takes finite values with positive probability. In other words, $\exists k_0$ such that $\nu(K = k_0) > 0$. But translation invariance of ν shows that $\nu(K = k)$ doesn't depend on k. Since $1 = \sum_k \nu(K = k)$, this is a contradiction, and we conclude that each $\mathcal{M} \cap \ell_k$ is non-empty a.s.

A similar argument shows that $\{x^{[2]} : x \in \mathcal{M} \cap \ell_k\}$ is a.s. unbounded below and above, for every k.

Suppose $z \notin \mathcal{M}$. Choose $x_0 \in \mathcal{M} \cap \ell_{z^{[1]}}$ so that $x_0^{[2]} < z^{[2]}$. Run the NW path from x_0 till it reaches some point x_1 with $x_1^{[2]} > z^{[2]}$. Then run NE paths from both x_0 and x_1. By Lemma 2.7

these paths meet at some point x_2. We have constructed two paths $x_0 \to x_1 \to x_2$ and $x_0 \to x_2$ in $\mathcal{C}_{x_0}$ which completely enclose z between them. Moreover $x_0 \in \mathcal{M}$ implies that $\mathcal{B}_{x_0} \supset \mathcal{M}_{x_0} = \mathcal{M}$ is infinite, so $\mathcal{B}_{x_0} = \mathbb{Z}^2$. So $\mathcal{M} = \mathcal{M}_{x_0} = \mathcal{C}_{x_0}$, and thus all vertices of the paths are in $\mathcal{M}$. Since the paths enclose z, they therefore enclose the connected component of $\mathbb{Z}^2 \setminus \mathcal{M}$ that contains z. Thus this component is finite. Since $z \notin \mathcal{M}$ was arbitrary, we've established that $\mathcal{M}$ is giant.

A similar argument works if we assume the hypotheses of Proposition 3.10 instead. We will just sketch the last step. For $z \notin \mathcal{M}$, find x_0 and x_1 in $\mathcal{M}$ with $x_0^{[1]} = x_1^{[1]} = z^{[1]}$ and $x_0^{[2]} < z^{[2]} < x_1^{[2]}$. Run NW paths from x_0 and x_1 till they meet at a point x_2. Run the SE paths from x_0 and x_1 till they meet at a point x_3. The four paths $x_0 \to x_2$, $x_1 \to x_2$, $x_0 \to x_3$, and $x_1 \to x_3$ all lie in $\mathcal{M}$ and enclose z between them. Thus they also enclose the connected component of $\mathbb{Z}^2 \setminus \mathcal{M}$ containing z, which must therefore be finite. □

In other words, having $\mathcal{B}_y = \mathbb{Z}^2$ is quite close to having $\mathcal{M}_y$ be a giant $\mathcal{M}$-component. Under regularity conditions on μ they are equivalent, but in principle having $\mathcal{B}_y = \mathbb{Z}^2$ could be weaker. We conjecture that this is not the case, assuming only (2.1).

Together with Corollary 3.11 this immediately yields the following result.

Corollary 4.12. *Consider the model* $(\leftrightarrow\updownarrow)$ *of Example 1.9. For each* $0 < p < 1$*,* $\mathcal{G}$ *has a giant* $\mathcal{M}$*-component almost surely.*

We could draw similar conclusions for the models $(\uparrow\!\!\rightarrow\ \leftarrow\!\!\downarrow)$ and $(\overset{\leftrightarrow}{\downarrow}\ \uparrow)$ of Examples 1.8 and 1.10 via Theorems 3.14 and 3.15. We do not record them at this point, in view of the quantitative versions we will establish below (Theorems 4.16 and 4.15). But note that this gives that giant $\mathcal{M}$ components exist for p in a neighbourhood of $1/2$. This qualitative conclusion was not obvious for Example 1.8 (to the authors at any rate), prior to conducting this work. Another plausible scenario would have been that giant $\mathcal{M}$ components only exist for $p = 1/2$, with the sizes of the $\mathcal{M}_x$ increasing without bound as $p \to 1/2$. As we have seen, this turns out not to be the case.

Note that the regularity hypotheses of Corollary 3.12 do not rule out the possibility that there exist infinite $\mathcal{M}_x$ which fail to be giant components. For example, this is the case with the following example, which satisfies the hypotheses of Corollary 3.12 if we rotate it by 180°:

Example 4.13. $(\overset{\leftrightarrow}{\downarrow}\ \leftrightarrow)$*:* $\mu(\{\rightarrow, \downarrow, \leftarrow\}) = p$ *and* $\mu(\{\rightarrow, \leftarrow\}) = 1 - p$*, where* $0 < p < 1$*. Then* $\mathcal{M}_x$ *is the horizontal line containing* x*.*

We do not believe that there are non-trivial examples of this behaviour, and hope to address this issue in subsequent work.

The rest of this section will be spent giving elementary arguments that giant $\mathcal{M}$ components exist in several models, for p's in various concrete intervals. Because of the structural results derived earlier, this automatically provides bounds on the critical percolation values $p_c^{\leftarrow\nwarrow\uparrow}$ and $p_c^{\leftarrow\nwarrow\swarrow\updownarrow}$. As far as we can tell, these bounds improve on what is in the literature (see (3.1) and (3.2)), but the main purpose is to understand what can be deduced in an elementary way using the approach through degenerate environments.

We note that results such as those of Burton and Keane [3] do not apply to many of our models, even though ν is stationary. The reason is that $\mathcal{M}$ need not be of "finite energy" in the sense given there. For example, in the (↑→ ←↓) model of Example 1.8, consider a configuration in which $(i,j) \in \mathcal{M}$ for $i=0$ and $j=0,1,2$ but $(i,j) \notin \mathcal{M}$ for $i=1,2$ and $j=0,1,2$. Then we cannot make a local modification that adds $(1,1)$ to $\mathcal{M}$ while leaving the other 8 given vertices unchanged. Such a change would require $\mathcal{G}_{(1,1)} =$ ←↓ in order to connect $(1,1)$ back to $\mathcal{M}$, and therefore $\mathcal{G}_{(1,0)} =$ ←↓ in order that $(1,0)$ not communicate directly with $(1,1)$. But that forces $(1,0)$ to connect to $\mathcal{M}$ via $(0,0)$ and from $\mathcal{M}$ via $(1,1)$, so that $(1,0) \in \mathcal{M}$.

We will use the notion of a *ring*, by which we mean a set of vertices R which can be enumerated as a closed path $x_0x_1x_2\dots x_N$ such that $x_0 = x_N$ and $x_{n+1} - x_n \in \mathcal{G}_{x_n}$ for $0 \le n < N$. The idea will be to construct, for each $M \ge 1$, a ring R_M that encloses the box $[-M,M]^2$. If $\mathcal{B}_x$ is infinite, then whenever $M > |x^{[1]}| \vee |x^{[2]}|$ we have $x \in [-M,M]^2$, so $\mathcal{B}_x$ intersects R_M. This implies that $R_M \subset \mathcal{B}_x$. Likewise $\mathcal{C}_x$ is infinite (assuming (2.1)), so intersects R_M, from which it follows that $R_M \subset \mathcal{C}_x$. Thus $R_M \subset \mathcal{M}_x$ for all sufficiently large M, implying that $\mathcal{M}_x$ is infinite. Moreover, the construction implies that all connected components of $\mathbb{Z}^2 \setminus \mathcal{M}_x$ are contained within some R_M, so must be finite. In other words, we have proved the following:

Lemma 4.14. *Assume $d = 2$ and (2.1). Assume further that rings R_M can be constructed enclosing arbitrarily large boxes $[-M,M]^2$. Then $\mathcal{M}_x$ is a giant $\mathcal{M}$ component whenever $\mathcal{B}_x$ is infinite.*

An easy application is to give an alternate proof of the existence of a giant $\mathcal{M}$ component in the (↔↕) model:

Alternate proof of Corollary 4.12. Let $M \ge 1$. Build a spiral path from the following pieces.

- Follow a SW path from $y_0 = (-M, M)$ till it reaches a point y_1 with $y_1^{[2]} = -M$.
- Then follow a SE path till it reaches a y_2 with $y_2^{[1]} = M$.
- Then follow a NE path till it reaches a y_3 with $y_3^{[2]} = M$.
- Then follow a NW path till it reaches a y_4 with $y_4^{[1]} = -M$.
- Then follow a SW path till it reaches a y_5 with $y_5^{[2]} = -M$.
- Then follow a SE path till it reaches a y_6 with $y_6^{[1]} = M$.
- Then follow a NE path till it reaches a y_7 with $y_7^{[2]} = M$.

It is possible that this spiral closes in on itself, in which case we've produced the desired ring. If it doesn't, suppose there is a point x in $[-M,M]^2$ with $\mathcal{B}_x$ infinite. Such a point will exist for large M by Proposition 3.6. In fact, by (c) of that result there is an infinite path leading into x, which must cross our spiral at some point z between y_4 and y_7, and then again at some point z' between y_0 and y_3. Following the spiral from z' around to z, and then moving from z to z' along the path in $\mathcal{B}_x$ produces the ring R_M. Now apply Lemma 4.14. □

Theorem 4.15. *Consider the model (↔↓ ↑) of Example 1.10. If $p^3 - p^2 + 2p - 1 > 0$ and $p < 1$ then $\mathcal{G}$ has a giant $\mathcal{M}$-component. In consequence, $p_c^{\nwarrow\leftarrow\swarrow\updownarrow} \ge 0.4311$*

Proof. We will again show that there is a giant $\mathcal{M}$-component by constructing arbitrarily large rings R_M. Our network contains the network ($\leftrightarrow\uparrow$) of Example 3.4, so by Proposition 3.6 there is an x such that $\mathcal{B}_x$ contains a semi-infinite path $\dots x_2 x_1 x_0$ to x such that $x^{[2]}$ is monotone. Without loss of generality, we take this $x = o$.

As in the alternate proof of Corollary 4.12, we may construct NE and NW paths in our network from arbitrary initial vertices. But we will also need paths that play the role of SE or SW paths in that proof. Here these will be paths that only move SE on average, or SW on average. So define the *SEoA* path from a $\overset{\leftrightarrow}{\downarrow}$ vertex to go $\downarrow$ if this leads to another $\overset{\leftrightarrow}{\downarrow}$ vertex, and otherwise to go $\rightarrow$. From a $\uparrow$ vertex, the path of course goes $\uparrow$. We define a *SWoA* path similarly. By construction, the *SEoA* path never takes a west step and the *SWoA* path never takes an east step. Our first task is to see when these paths actually do – on average – move in the desired direction.

Let z be the initial point of the SEoA path, and let W be the vertical distance travelled before moving sideways. So $W = -k \leq 0$ means that z as well as the k points directly below it are $\overset{\leftrightarrow}{\downarrow}$, while the point below them is $\uparrow$. Likewise, $W = k \geq 1$ means that z as well as the $k-1$ points directly above it are $\uparrow$, while the point above them is $\overset{\leftrightarrow}{\downarrow}$. Thus

$$\mathbb{E}[W] = -\sum_{k\geq 0} kp^{k+1}(1-p) + \sum_{k\geq 1} kp(1-p)^k = -\frac{p^2}{1-p} + \frac{1-p}{p} = -\frac{p^3 - p^2 + 2p - 1}{p(1-p)}.$$

In particular, if $p^3 - p^2 + 2p - 1 > 0$, then the SEoA paths (resp. SWoA paths) drift SE (resp. SW) on average.

So assume $p^3 - p^2 + 2p - 1 > 0$, and construct the ring as follows. Let $z^i = z_0^i z_1^i \dots$ be the SEoA path starting from $(-i, M)$ and ending at (M, k) for some k. Note that two such paths coalesce if/when they meet. Because $\mathbb{E}[W] < 0$, the probability that $[-M, M]^2$ lies entirely above this path converges to 1 as $i \to \infty$. So we may almost surely find an i so that this is this the case. In fact, we may find an increasing sequence $i(0), i(1), \dots$ such that $z^{i(k)}$ lies entirely above $z^{i(k+1)}$ for $k \geq 0$, and $[-M, M]^2$ lies entirely above each $z^{i(k)}$.

We may now build a spiral path as follows:

- From $y_1 = z_0^{i(0)}$ (which has $y_1^{[2]} = M$), follow the SEoA path till it reaches a point y_2 with $y_2^{[1]} = M$. By construction, $[-M, M]^2$ lies above this path.
- From y_2 follow the NE path till it reaches a point y_3 with $y_3^{[2]} = M$.
- From y_3 follow the NW path till it reaches a point y_4 with $y_4^{[1]} = -M$.
- From y_4 follow the SWoA path till it hits some path $z^{i(k)}$ at a point y_5. It must do so because eventually it lies below the line $x^{[2]} = M$, so $z^{i(k)}$ will cross it for k sufficiently large.
- From y_5 follow the SEoA path $z^{i(k)}$ till it reaches a point y_6 with $y_6^{[1]} = M$.
- From y_6 follow the NE path till it reaches a point y_7 with $y_7^{[2]} = M$.

It is possible that this spiral closes in on itself, in which case we've produced the desired ring. If it doesn't, recall that we have an infinite path $\dots x_2 x_1 x_0$ leading to o in $\mathcal{B}_o$, with $x_i^{[2]}$ monotone decreasing. This must cross our spiral at some point x' between y_4 and y_7, and then again at some point x'' between y_1 and y_3. Following the spiral from x'' around to x', and then moving from x' to x'' along the path in $\mathcal{B}_o$ produces the ring R_M. Now apply Lemma 4.14.

The estimate on $p_c^{⤡}$ comes from computing the unique root of the increasing function $p^3 - p^2 + 2p - 1$. By Lemma 4.10 we will have some $\mathcal{B}_x = \mathbb{Z}^2$ for p above this root, and can then apply Theorem 3.15 ☐

Theorem 4.16. *Consider the model (↑→ ←↓) of Example 1.8. If $p^3 + 2p - 1 \geq 0$ and $(1-p)^3 + 2(1-p) - 1 \geq 0$ then $\mathcal{G}$ has a giant $\mathcal{M}$-component almost surely. In consequence $p_c^{⤢} \geq 0.5466$.*

Proof. This network includes the network (↑→, ←) of Example 3.7, so by Proposition 3.8 there are infinite $\mathcal{B}$'s. In fact, let $\mathcal{N}_y \subset \mathcal{B}_y$ denote the cluster of points from which y can be reached, using steps $\uparrow, \leftarrow, \rightarrow$. Let $L_n(y)$ and $U_n(y)$ be the infimum and supremum of l with $(l, y^{[2]} - n) \in \mathcal{N}_y$. It follows from Proposition 3.8 that $A_y = \{|L_n(y)| < \infty \forall n\}$ has probability > 0. Without loss of generality we (for now) take $y = o$ and write $L_n = L_n(o)$.

Consider the SE path from some point z lying to the left of $\mathcal{N}_o$. Our first task will be to determine what choices of p imply that, a.s. on the event A_o, this path hits $\mathcal{N}_o$. Without loss of generality, $z = (w, 0)$ for some $w < L_0$, and $\mathcal{G}_z =$ ←↓.

Recall that on the event A_o, L_n agrees with the path of a random walk L_n^0. We may construct L_n^0 as follows: From $(L_n^0, -n)$ look down one vertex. If $\mathcal{G}_{(L_n^0, -(n+1))} =$ ↑→ then $L_{n+1}^0 \leq L_n^0$, and is in fact the smallest value such that $\mathcal{G}_{(j,-(n+1))} =$ ↑→ for $L_{n+1}^0 \leq j \leq L_n^0$. On the other hand, if $\mathcal{G}_{(L_n^0, -(n+1))} = \leftarrow$, then $L_{n+1}^0 > L_n^0$, and is in fact the smallest $j > L_n^0$ such that $\mathcal{G}_{(j,-(n+1))} =$ ↑→. As observed in the proof of Proposition 3.8,

$$\nu(L_{n+1}^0 = l + j \mid L_n^0 = l) = \begin{cases} (1-p)p^{1-j}, & j \leq 0 \\ p(1-p)^j, & j > 0 \end{cases}, \qquad \mathbb{E}[\Delta L_n^0] = \frac{1-p}{p} - \frac{p^2}{1-p}.$$

Set $W_0 = w$. Let $W_1, W_2, \ldots$ be the first coordinates of successive vertices at which the SE path moves downward. In other words, the downwards steps are from $(W_n, -n)$ to $(W_n, -(n+1))$. Our object is to show that with probability 1 there exists an n with $W_n \geq L_n^0$.

W_n is itself a random walk, with $\nu(W_{n+1} = l + j \mid W_n = l) = (1-p)p^j$ for $j \geq 0$, and $\mathbb{E}[\Delta W_n] = p/(1-p)$. We see that $\mathbb{E}[\Delta W_n] \geq \mathbb{E}[\Delta L_n^0]$ provided

$$\frac{p}{1-p} \geq \frac{1-p}{p} - \frac{p^2}{1-p} \quad \iff \quad p^3 + 2p - 1 \geq 0.$$

We assume, in what follows, that this inequality holds. In particular, this is true for $p \geq 0.4534$; If W_n and L_n^0 were independent, the desired conclusion would follow immediately. Because these walks are actually not quite independent, we need to be slightly more careful.

When $W_n = j < l = L_n^0$, we have

$$\nu(W_{n+1} = j', L_{n+1}^0 = l' \mid W_n = j, L_n^0 = l) = \nu(W_{n+1} = j' \mid W_n = j) \cdot \nu(L_{n+1}^0 = l' \mid L_n^0 = l),$$

provided $j' < l$ and $j' < l' - 1$, since the two events in question depend on disjoint parts of the environment. Let $C(j, l)$ be the set of (j', l') which violate the above condition. Then we can describe the evolution of the Markov chain (W_n, L_n^0) as follows: From $(W_n, L_n^0) = (j, l)$ propose a move to a (j', l') chosen based on W_{n+1} and L_{n+1}^0 evolving independently. If $(j', l') \notin C(j, l)$ the move is accepted. Otherwise the move is rejected, and replaced by a move to some point of $C(j, l)$ chosen according to the required law. The fact that $\mathbb{E}[\Delta W_n] \geq \mathbb{E}[\Delta L_n^0]$ implies that with probability 1, this chain will eventually encounter a rejected move.

What moves in $C(j, l)$ can replace a rejected move? There are three types – either $j' \geq l'$ (if $\mathcal{G}_{(i,-(n+1))} = \uparrow\!\rightarrow$ for $j \leq i \leq l$), or $j' + 1 = l' \leq l$ (if $\mathcal{G}_{(l,-(n+1))} = \uparrow\!\rightarrow$ and exactly one $j \leq i < l$ has $\mathcal{G}_{(i,-(n+1))} = \leftarrow\!\downarrow$), or $j' = l < l'$ (if $\mathcal{G}_{(l,-(n+1))} = \leftarrow\!\downarrow$ and all $j \leq i < l$ have $\mathcal{G}_{(i,-(n+1))} = \uparrow\!\rightarrow$). In the first case, $W_{n+1} \geq L_{n+1}^0$ and the paths cross. In the second case, there is a high probability of crossing on the next step. It is then an easy calculation to show that there is an $\epsilon > 0$ such that

$$\nu(W_{n+1} \geq L_{n+1}^0 \text{ or } W_{n+2} \geq L_{n+2}^0 \mid W_n = j, L_n^0 = l) \geq \epsilon \nu((W_{n+1}, L_{n+1}^0) \in C(j, l) \mid W_n = j, L_n^0 = l).$$

Thus after at most finitely many rejected moves, we will eventually find one leading to the desired intersection. We have therefore shown that with probability 1, there exists an n with $W_n \geq L_n^0$, as required.

Suppose that $p^3 + 2p - 1 \geq 0$. We claim that with probability 1 there is a $j > M$ such that $\mathcal{N}_{(j,M)}$ is infinite and $[-M, M]^2$ lies to the left of $\mathcal{N}_{(j,M)}$. To see this, let $\epsilon = \nu(|\mathcal{N}_{(j,M)}| = \infty) > 0$. Because $\mathbb{E}[\Delta L_n^0] > -\infty$ there is a $j_0 > M$ such that

$$\nu(|\mathcal{N}_{(j_0,M)}| = \infty \text{ and } [-M, M]^2 \text{ lies to the left of } \mathcal{N}_{(j_0,M)}) \geq \epsilon/2.$$

Search down from (j_0, M) to see if this event occurs. Either the search succeeds, or it fails after examining a finite number k_0 of vertices. If it fails, we may likewise find a $j_1 > j_0$ such that

$$\nu(|\mathcal{N}_{(j_1,M)}| = \infty \text{ and } [-(M + j_0 + k_0), M + j_0 + k_0]^2 \text{ is left of } \mathcal{N}_{(j_1,M)}) \geq \epsilon/2.$$

Search down from (j_1, M) to see if this event occurs. Note that this search will not involve any of the k_0 vertices already examined, so it succeeds or fails independently of what has come before. Either the search succeeds, or it fails after examining a finite number k_1 of vertices, and the process can now be repeated. Eventually the search must succeed, giving the claimed value j.

Suppose also that $(1 - p)^3 + 2(1 - p) - 1 \geq 0$. Define $\tilde{\mathcal{N}}_y$ as above, but using moves $\downarrow, \leftarrow, \rightarrow$. Then by symmetry, a similar argument shows that there is a $\tilde{j} < -M$ such that $\tilde{\mathcal{N}}_{(\tilde{j},-M)}$ is infinite, and $[-M, M]^2$ lies to the right of $\tilde{\mathcal{N}}_{(\tilde{j},-M)}$. As well, NW paths from the right of $\tilde{\mathcal{N}}_{(\tilde{j},-M)}$ will a.s. intersect this set.

Finally, we can construct our ring R_M. Starting from $(\tilde{j}, -M)$ follow the SE path till it intersects $\mathcal{N}_{(j,M)}$. By definition of $\mathcal{N}_{(j,M)}$ we may then follow a path in this set up to (j, M), and by construction $[-M, M]^2$ lies to the left of this path. Now follow a NW path till it intersects $\tilde{\mathcal{N}}_{(\tilde{j},-M)}$. By definition we may follow a path in this set up to $(\tilde{j}, -M)$, which closes up the desired ring. Now apply Lemma 4.14.

The estimate on $p_c^{\nwarrow\!\searrow}$ comes from computing the unique root r of the increasing function $p^3 + 2p - 1$. We will have some $\mathcal{B}_x = \mathbb{Z}^2$ for p between r and $1 - r$, and can then apply Theorem 3.14. $\square$

In the case of Example 3.16, by Corollary 4.11, the models $(\leftarrow\!\downarrow\!\rightarrow \; \uparrow\!\rightarrow)$, $(\leftrightarrow\!\updownarrow \; \uparrow)$, and $(\leftrightarrow\!\updownarrow \; \uparrow\!\rightarrow)$ all exhibit giant $\mathcal{M}$-components for some p but not for others. When they do, we already know that

$\mathcal{B}_x = \mathbb{Z}^2$ for every x. So in that case also $\mathcal{M}_x = \mathbb{Z}^2$ for every x. We have attempted to estimate $p_c^{\leftarrow\nwarrow\uparrow}$ and $p_c^{\leftarrow\nwarrow\swarrow\updownarrow}$ using these models, but did not obtain any improvement on the values given earlier. The arguments were slightly simpler than those given, but we felt this was offset by the added complexity of the models.

5 Model summary

Here we summarize the results of earlier sections as applied to 2-dimensional 2-valued environments. In each case the first possibility is assumed to have probability $p \in (0,1)$, and the second $1-p$.

Notes to Table 2

(i) is infinite with probability $> 0 \Longleftrightarrow \mathcal{C}_o$ is.

(ii) Phase transition: $\mathcal{C}_o$ is finite for $p < p_c$, and ∞ with probability > 0 if $p > p_c$.

(iii) All $\mathcal{C}_x$ intersect.

(iv) All infinite $\mathcal{B}_x$ intersect.

(v) [1] All infinite $\mathcal{B}_x$ are blocked (above or below)

[2] All infinite $\mathcal{B}_x$ equal $\mathbb{Z}^2$

(vi) [1] $\exists!$ giant $\mathcal{M}$-component.

[2] All $\mathcal{M}_x = \mathbb{Z}^2$.

[3] There are multiple infinite $\mathcal{M}_x$.

Model	$\nu(\lvert\mathcal{C}_o\rvert=\infty)$	$\nu(\lvert\mathcal{B}_o\rvert=\infty)$	$\nu(\lvert\mathcal{M}_o\rvert=\infty)$	Notes	E.g.
$\uparrow\ \cdot$	0	0	0	1-dimensional	
$\leftrightarrow\ \cdot$	0	0	0	1-dimensional	
$\{\uparrow\rightarrow\}\ \cdot$	$>0(p>p_c^{\{\uparrow\rightarrow\}})$	*see (i)*	0	oriented site percolation; *ii*	1.3
$\{\leftrightarrow\downarrow\}\ \cdot$	$>0(p>p_c^{\{\leftrightarrow\downarrow\}})$	*see (i)*	0	partially-oriented site perc.; *ii*	2.10
$\{\leftrightarrow\updownarrow\}\ \cdot$	$>0(p>p_c^{\{\leftrightarrow\updownarrow\}})$	*see (i)*	*see (i)*	site percolation; *ii*	1.2
$\uparrow\ \rightarrow$	1	0	0	coalescing RW; *iii*	1.5
$\uparrow\ \downarrow$	0	0	0	1-dimensional	2.2
$\leftrightarrow\ \uparrow$	1	>0	0	*iii*, *iv*, v^1	3.5
$\leftrightarrow\ \rightarrow$	1	1	0	1-dimensional	2.8
$\leftrightarrow\ \updownarrow$	1	>0	>0	*iii*, *iv*, v^2, vi^1	1.9
$\{\uparrow\rightarrow\}\ \uparrow$	1	1	0	*iii*, *iv*	3.4
$\{\uparrow\rightarrow\}\ \{\leftarrow\uparrow\}$	1	1	0	*iii*, *iv*, v^1	1.1
$\{\uparrow\rightarrow\}\ \leftrightarrow$	1	1	0	*iii*, *iv*, v^1	
$\{\uparrow\rightarrow\}\ \leftarrow$	1	>0	0	*iii*, *iv*	3.7
$\{\uparrow\rightarrow\}\ \{\leftarrow\downarrow\}$	1	>0	phase trans.	*iii*, *iv*, v^1 $(p<1-p_c^{\{\nwarrow\leftarrow\uparrow\}}$ or $>p_c^{\{\nwarrow\leftarrow\uparrow\}})$ *iii*, *iv*, v^2, vi^1 $(1-p_c^{\{\nwarrow\leftarrow\uparrow\}}<p<p_c^{\{\nwarrow\leftarrow\uparrow\}})$	1.8
$\{\leftrightarrow\downarrow\}\ \downarrow$	1	1	0	*iii*, *iv*, v^1	
$\{\leftrightarrow\downarrow\}\ \rightarrow$	1	1	0	*iii*, *iv*	
$\{\leftrightarrow\downarrow\}\ \uparrow$	1	>0	phase trans.	*iii*, *iv*, v^1 $(p<1-p_c^{\{\nwarrow\leftarrow\swarrow\updownarrow\}})$ *iii*, *iv*, v^2, vi^1 $(p>1-p_c^{\{\nwarrow\leftarrow\swarrow\updownarrow\}})$	1.10
$\{\leftrightarrow\downarrow\}\ \leftrightarrow$	1	1	1	*iii*, *iv*, v^1, vi^3	4.13
$\{\leftrightarrow\downarrow\}\ \updownarrow$	1	1	1	*iii*, *iv*, v^2, vi^2	
$\{\leftrightarrow\downarrow\}\ \{\uparrow\rightarrow\}$	1	1	phase trans.	*iii*, *iv*, v^1 $(p<1-p_c^{\{\nwarrow\leftarrow\uparrow\}})$ *iii*, *iv*, v^2, vi^2 $(p>1-p_c^{\{\nwarrow\leftarrow\uparrow\}})$	3.16a
$\{\leftrightarrow\downarrow\}\ \{\leftarrow\downarrow\}$	1	1	0	*iii*, *iv*, v^1	
$\{\leftrightarrow\downarrow\}\ \{\updownarrow\rightarrow\}$	1	1	1	*iii*, *iv*, v^2, vi^2	
$\{\leftrightarrow\downarrow\}\ \{\leftrightarrow\uparrow\}$	1	1	1	*iii*, *iv*, v^2, vi^2	4.8
$\{\leftrightarrow\updownarrow\}\ \uparrow$	1	1	phase trans.	*iii*, *iv*, v^1 $(p<1-p_c^{\{\nwarrow\leftarrow\swarrow\updownarrow\}})$ *iii*, *iv*, v^2, vi^2 $(p>1-p_c^{\{\nwarrow\leftarrow\swarrow\updownarrow\}})$	3.16c
$\{\leftrightarrow\updownarrow\}\ \{\uparrow\rightarrow\}$	1	1	phase trans.	*iii*, *iv*, v^1 $(p<1-p_c^{\{\nwarrow\leftarrow\uparrow\}})$ *iii*, *iv*, v^2, vi^2 $(p>1-p_c^{\{\nwarrow\leftarrow\uparrow\}})$	3.16b
$\{\leftrightarrow\updownarrow\}\ \leftrightarrow$	1	1	1	*iii*, *iv*, v^2, vi^2	
$\{\leftrightarrow\updownarrow\}\ \{\leftrightarrow\downarrow\}$	1	1	1	*iii*, *iv*, v^2, vi^2	

Table 2: Table of connectivity results for 2-dimensional 2-valued degenerate random environments.

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