The survival probability in high dimensions

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Abstract

In this paper, we investigate the *survival probability* θ_n in high dimensional statistical physical models. We prove that when the r-point functions scale to those of the canonical measure of super-Brownian motion, and when a certain self-repellence condition is satisfied, then $n\theta_n \to 2/(AV)$, where A is the asymptotic expected number of particles alive at time n, and V is the vertex factor of the model. Our results apply to spread-out lattice trees above 8 dimensions, spread-out oriented percolation above $4+1$ dimensions, and the spread-out contact process above $4+1$ dimensions. In the case of oriented percolation, this reproves a result by the first author, den Hollander and Slade (that was proved using heavy lace expansion arguments), at the cost of losing explicit error estimates. Our proof is based on simple weak convergence arguments.

1 Introduction and results

Consider a Galton-Watson branching process with offspring distribution (supported on $\mathbb{Z}_+$) that has mean 1 and variance γ, starting from a single initial particle. Let S_n be the event that the process survives until time n, and $\theta_n = \mathbb{P}(S_n)$. It is well known (see e.g. [25, Theorem II.1.1.]) that $n\theta_n \to 2/\gamma$ as $n \to \infty$, and that the population size N_n at time n is such that, conditionally on $N_n > 0$, the random variable $n^{-1}N_n$ converges weakly to a random variable Y that has an exponential distribution with mean $\gamma/2$. Embedding the branching process into $\mathbb{Z}^d$, with the initial particle located at the origin, $0 \in \mathbb{Z}^d$, and where the offspring of any given particle are independently located at neighbors of that particle in $\mathbb{Z}^d$, we obtain a branching random walk. Since multiple occupancy can occur, the state of this process at time n is best described by a measure, where the measure of any subset of $\mathbb{R}^d$ is the number of particles of generation n located in that set. With appropriate rescaling of space, time, mass (associated to each particle), and of the underlying law, we obtain a sequence of finite (no longer probability) measures μ_n that converge weakly to a measure $\mathbb{N}_0$ on the space of measure valued paths $(X_t)_{t\geq 0}$ that survive for positive time, i.e. $S \equiv \inf\{t > 0 : X_t(1) = 0\} > 0$ (where $X_t(f) \equiv \int f dX_t$). The measure $\mathbb{N}_0$ is called the canonical measure of super-Brownian motion and is σ-finite, with $\mathbb{N}_0(S > \varepsilon) = 2/\varepsilon$ for every $\varepsilon > 0$. The notion of weak convergence is defined with respect to the finite measures $\mathbb{N}_0^{\varepsilon}(\cdot) \equiv \mathbb{N}_0(\cdot, S > \varepsilon)$ (see e.g. [20]), and in particular $n\theta_{\lfloor nt \rfloor} \to \gamma^{-1}\mathbb{N}_0(S > t) = 2/(\gamma t)$.

In this paper we work with general models in statistical mechanics that converge (or are conjectured to converge) to super-Brownian motion (SBM) in some sense. Let $\mathbb{P}$ denote the probability measure describing the law of our model. We start by introducing some notation in terms of which we can define our conditions as well as results. All our models will have a notion of *intrinsic distance*, in which $x \xrightarrow{n} y$ means that x is connected to y in a path of length n. Then, for $\vec{x} \in \mathbb{Z}^{d(r-1)}$ and $\vec{n} \in \mathbb{N}^{r-1}$, we let

$$t_{\vec{n}}^{(r)}(\vec{x}) = \mathbb{P}(0 \xrightarrow{n_i} x_i \forall i = 1, \ldots, r-1) \tag{1.1}$$

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denote the r-point functions in the model, for $\vec{k} = (k_1, \ldots, k_{r-1}) \in ([-\pi, \pi]^d)^{r-1}$, we let

$$\widehat{t}^{(r)}_{\vec{n}}(\vec{k}) = \sum_{\vec{x} \in \mathbb{Z}^{d(r-1)}} \mathrm{e}^{i\vec{k}\cdot\vec{x}} t^{(r)}_{\vec{n}}(\vec{x}) \tag{1.2}$$

denote the Fourier transform of the r-point function, and

$$\theta_n = \mathbb{P}(\exists x \in \mathbb{Z}^d \colon 0 \xrightarrow{n} x) \tag{1.3}$$

the survival probability. Let $A_n = \{x \colon 0 \xrightarrow{n} x\}$, $N_n = \#\{x \colon 0 \xrightarrow{n} x\}$, and $S_n = \{N_n > 0\} = \{A_n \neq \varnothing\}$, so that $\theta_n = \mathbb{P}(S_n)$. In some models time is continuous, in which case we replace $n \in \mathbb{N}$ by $t \in [0, \infty)$. Also, in what follows, when the underlying model is defined in discrete time, we define $n\vec{t}$ to be the vector $(\lfloor nt_1 \rfloor, \ldots, \lfloor nt_r \rfloor)$.

In this paper, we investigate the asymptotics of the survival probability, assuming the asymptotic behavior of the r-point functions. These results apply to (a) lattice trees, where the necessary bounds were proved in [4, 5, 18]; (b) oriented percolation, where the necessary bounds were proved in [2, 17], see also [23, 24]; and (c) the contact process, where the necessary bounds were proved in [1, 14, 15].

In particular, when the allowed connections are sufficiently spread out, e.g. where all vertices within distance $L \gg 1$ of a vertex are considered to be neighbors of that vertex, the following condition holds as a theorem for each of these models, above their respective critical dimensions:

Condition 1.1 (Convergence of the r-point functions)**.** *There exist constants $A, V, v > 0$ all depending on L such that*

$$\frac{1}{A(VA^2n)^{r-2}} \widehat{t}^{(r)}_{n\vec{t}}\left(\frac{\vec{k}}{\sqrt{vn}}\right) \to \widehat{M}^{(r-1)}_{\vec{t}}(\vec{k}), \quad \text{as } n \to \infty, \tag{1.4}$$

where the quantities $\widehat{M}^{(r-1)}_{\vec{t}}(\vec{k})$ are the Fourier transforms of the moment measures of the canonical measure of SBM. In particular, $\widehat{M}^{(r-1)}_{t\vec{1}_{r-1}}(\vec{0}) = t^{r-2}2^{-(r-2)}(r-1)!$.

Condition 1.1 can be rephrased as

$$\mathbb{E}_{\mu_n}\left[\prod_{j=1}^{r-1} X^{(n)}_{t_j}(\phi_{k_j})\right] \to \mathbb{E}_{\mathbb{N}_0}\left[\prod_{j=1}^{r-1} X_{t_j}(\phi_{k_j})\right], \tag{1.5}$$

where $\phi_{k_j}(x) = \mathrm{e}^{ik_j\cdot x}$ for $k_j \in \mathbb{R}^d$ and $x \in \mathbb{Z}^d$, and where

$$X^{(n)}_t(f) = \frac{1}{VA^2n} \sum_{x \in A_{nt}} f(x/\sqrt{vn}), \quad \text{and} \quad \mu_n(\cdot) = nVA\mathbb{P}(\cdot). \tag{1.6}$$

Thus, Condition 1.1 states that certain moment measures of the rescaled processes under the measure μ_n converge to those of the canonical measure of SBM.

Before stating our main result, we start by formulating two further conditions. Let

$$\mathcal{C}(0) = \{(x, n) \colon 0 \xrightarrow{n} x\} \tag{1.7}$$

denote the *oriented cluster* of $0 \in \mathbb{Z}^d$, i.e., all vertices $x \in \mathbb{Z}^d$ to which 0 is connected, and we let $|\mathcal{C}(0)|$ denote its size. In continuous-time models, we instead take

$$|\mathcal{C}(0)| = \int_0^\infty \#\{x \colon 0 \xrightarrow{t} x\} dt. \tag{1.8}$$

We make two central assumptions on our high-dimensional models:

Condition 1.2 (Cluster tail bound)**.** *There exists a constant $C_{\mathcal{C}}$ such that*

$$\mathbb{P}(|\mathcal{C}(0)| \geq k) \leq C_{\mathcal{C}}/\sqrt{k}. \tag{1.9}$$

Condition 1.3 (Self-repellent survival property)**.** *Let $\mathcal{F}_m$ be the σ-field generated by the vertices at distance at most m from 0, i.e. by $\{(x,n)\colon 0 \xrightarrow{n} x, n \leq m\}$. Then there exists a constant C_θ such that, with N_m equal to the number of x with $0 \xrightarrow{m} x$, almost surely for every stopping time $M \leq n$,*

$$\mathbb{P}(A_M \longrightarrow n \mid \mathcal{F}_M) \leq C_\theta N_M \theta_{n-M} \tag{1.10}$$

The cluster tail condition follows from the literature for all models under consideration. The self-repellent survival property in (1.10) turns out to be easy to check, and we shall do this below. Our main result is the following theorem:

Theorem 1.4. *When Conditions 1.1, 1.2 and 1.3 hold, as $n \to \infty$*

$$n\theta_n \to 2/(AV). \tag{1.11}$$

Consequently,

$$\mu_n(X_t^{(n)}(1) > 0) \to \mathbb{N}_0(X_t(1) > 0) = 2/t. \tag{1.12}$$

We remark that Theorem 1.4 also follows when, instead of Condition 1.3, we assume that $\theta_n \leq C/n$.

For oriented percolation, the result reproves a result from [12, 13] (but without the error estimates) in a relatively simple way. See also [21, 22, 27, 28] for related results on survival probabilities. Our set-up is rather general, so that in the future, it might be applicable to percolation and lattice animals as well.

The result in Theorem 1.4 is particularly important, since the combination of the convergence of the r-point functions as formulated in Condition 1.1 and Theorem 1.4, jointly with the results in [20], imply that $\{\mu_n\}_{n\geq 1}$ converge in the sense of *finite-dimensional distributions* to $\mathbb{N}_0$.

We now present some examples. All of the examples involve a function $D\colon \mathbb{Z}^d \to [0,1]$, with $\sum_{x\in\mathbb{Z}^d} D(x) = 1$ that obeys the properties of Assumption D in [16, Section 1.2] (whose precise form is not important for the present paper), together with [17, Equation (1.2)]. This assumption involves a parameter $L \in \mathbb{N}$, which serves to spread out the connections and which will be taken to be large.

Spread-out oriented percolation above $4+1$ dimensions. The spread-out oriented bond percolation model is defined as follows. Consider the graph with vertices $\mathbb{Z}^d \times \mathbb{Z}_+$ and with directed bonds $((x,n),(y,n+1))$, for $n \in \mathbb{Z}_+ = \{0,1,2,\ldots\}$ and $x,y \in \mathbb{Z}^d$. Let $p \in [0, \|D\|_\infty^{-1}]$, where $\|\cdot\|_\infty$ denotes the supremum norm, so that $pD(x) \leq 1$ for all $x \in \mathbb{Z}^d$. We associate to each directed bond $((x,n),(y,n+1))$ an independent random variable taking the value 1 with probability $pD(y-x)$ and the value 0 with probability $1 - pD(y-x)$. We say that a bond is *occupied* when the corresponding random variable is 1 and *vacant* when it is 0. The joint probability distribution of the bond variables will be denoted by $\mathbb{P}_p$, and the corresponding expectation by $\mathbb{E}_p$.

We say that (x,n) is *connected* to (y,m), and write $(x,n) \longrightarrow (y,m)$, if there is an oriented path from (x,n) to (y,m) consisting of occupied bonds. Note that this is only possible when $m \geq n$. By convention, (x,n) is connected to itself. We write $(x,n) \longrightarrow m$ if $m \geq n$ and there is a $y \in \mathbb{Z}^d$ such that $(x,n) \longrightarrow (y,m)$. The event $\{(0,0) \longrightarrow \infty\}$ is the event that $\{(0,0) \longrightarrow n\}$ occurs for all n. There is a critical threshold $p_c > 0$ such that the event $\{(0,0) \longrightarrow \infty\}$ has probability zero for $p < p_c$ and has positive probability for $p > p_c$. The survival probability at time n is defined by

$$\theta_n(p) = \mathbb{P}_p((0,0) \longrightarrow n), \tag{1.13}$$

and we let $\theta_n = \theta_n(p_c)$. General results of [3, 6] imply that $\lim_{n\to\infty} \theta_n = 0$.

Then, for $\mathbb{P} = \mathbb{P}_{p_c}$, Condition 1.1 is proved in [17]. Condition 1.2 holds by [2, 17, 23, 24], while Condition 1.3 follows from a union bound (i.e. $\mathbb{P}(\cup_{x\in A_M}\{x \to n\}|\mathcal{F}_M) \leq \sum_{x\in A_M} \mathbb{P}(x \to n|\mathcal{F}_M)$) and the strong Markov property.

Spread-out contact process above $4+1$ dimensions. We define the spread-out contact process as follows. Let $\mathcal{C}_t \subset \mathbb{Z}^d$ be the set of infected individuals at time $t \in \mathbb{R}_+$, and let $\mathcal{C}_0 = \{0\}$. An infected site x recovers in a small time interval $[t, t+\varepsilon]$ with probability $\varepsilon + o(\varepsilon)$ independently of t, where $o(\varepsilon)$ is a function that satisfies $\lim_{\varepsilon \to 0} o(\varepsilon)/\varepsilon = 0$. In other words, $x \in \mathcal{C}_t$ recovers at rate 1. A healthy site x gets infected, depending on the status of its neighbors, at rate $\lambda \sum_{y \in \mathcal{C}_t} D(x-y)$, where $\lambda \geq 0$ is the infection rate. We denote by $\mathbb{P}^\lambda$ the associated probability measure.

By an extension of the results in [3, 6] to the spread-out contact process, there exists a unique critical value $\lambda_c \in (0, \infty)$ such that

$$\theta(\lambda) \equiv \lim_{t\to\infty} \mathbb{P}^\lambda(\mathcal{C}_t \neq \varnothing) \begin{cases} = 0, & \text{if } \lambda \leq \lambda_c, \\ > 0, & \text{if } \lambda > \lambda_c, \end{cases} \tag{1.14}$$

and we define

$$\theta_t = \theta_t(\lambda_c) = \mathbb{P}^{\lambda_c}(\mathcal{C}_t \neq \varnothing). \tag{1.15}$$

Condition 1.1 is proved in [14, 15]. Condition 1.2 holds by [14, 15, 1, 26], while Condition 1.3 again follows from a union bound and the strong Markov property.

Spread-out lattice trees above 8 dimensions. A lattice tree is a finite connected set of lattice bonds (and their associated end vertices) containing no cycles. For fixed $z > 0$, every such tree $T \ni 0$ with bond set B is assigned a weight $W_z(T) = z^{|B|} \prod_{(x,y)\in B} D(y-x)$, and we define $\rho_z(x) = \sum_{T \ni 0,x} W_z(T)$. The radius of convergence z_c of $\sum_{x\in\mathbb{Z}^d} \rho_z(x)$ is finite. Let $W(\cdot) = W_{z_c}(\cdot)$ and $\rho = \rho_{z_c}(0)$. We define a probability measure on the (countable) set of lattice trees containing the origin by $\mathbb{P}(T) = \frac{W(T)}{\rho}$. Given a lattice tree $T \ni 0$, we define $A_n(T) = \{a_1, \ldots, a_{N_n}\}$ to be the (ordered) set of vertices in T of tree distance n from the origin under some arbitrary but fixed ordering of $\mathbb{Z}^d$.

Condition 1.1 is the main result in [19]. Condition 1.2 follows from the detailed asymptotics for $\mathbb{P}(|T| = n) \sim cn^{-3/2}$ proved in [4, 5]. We next check Condition 1.3, for which it is enough to show that the result holds a.s. for every deterministic time $m \leq n$. Letting T_m denote the tree up to tree distance m from the root, we have that $\mathbb{P}(A_m \longrightarrow n \mid T_m = \tau_m)$ is equal to

$$\frac{W(\tau_m)}{\sum_{T:\, T_m=\tau_m} W(T)} \sum_{R_1 \ni a_1} \cdots \sum_{R_{N_m} \ni a_{N_m}} \prod_{i=1}^{N_m} W(R_i) \mathbb{1}_{\{R_i \text{ avoid each other and } \tau_m\}} \mathbb{1}_{\{\cup_j R_j \text{ survives at least until } n-m\}},$$

where $\sum_{R\ni a}$ is a sum over lattice trees R containing $a \in \mathbb{Z}^d$, and we recall that $A_m = \{a_1, \ldots, a_{N_m}\}$.

The final indicator function is bounded above by $\sum_j \mathbb{1}_{\{\mathcal{S}_{R_j} \geq n-m\}}$, where $\mathcal{S}_T$ is the survival time of T. By taking the sum over j outside and dropping the restriction that R_j avoids other R_i and τ_m, this is bounded above by

$$\begin{aligned} &\sum_{j=1}^{N_m} \sum_{R_j \ni a_j} W(R_j) I_{\{\mathcal{S}_{R_j} \geq n-m\}} \Bigg[\frac{W(\tau_m)}{\sum_{T:\, T_m=\tau_m} W(T)} \\ &\times \sum_{R_1\ni a_1} \cdots \sum_{R_{j-1}\ni a_{j-1}} \sum_{R_{j+1}\ni a_{j+1}} \cdots \sum_{R_{N_m}\ni a_{N_m}} \prod_{i\neq j} W(R_i) \mathbb{1}_{\{R_i, i\neq j \text{ avoid each other and } \tau_m\}} \Bigg] \\ &\leq \sum_{j=1}^{N_m} \sum_{R_j\ni a_j} W(R_j) \mathbb{1}_{\{\mathcal{S}_{R_j} \geq n-m\}} = N_m \rho \theta_{n-m}, \end{aligned} \tag{1.16}$$

where we have used the fact that the interaction term makes the graph $\tau_m \cup_{i\neq j} R_i$ a lattice tree T with $T_m = \tau_m$, and weight $W(T) = W(\tau_m) \prod_{i\neq j} W(R_i)$, so the numerator in brackets is no more than the denominator. This verifies Condition 1.3.

Spread-out percolation above 6 dimensions. Let $p \in [0, \|D\|_\infty^{-1}]$ be a parameter. We declare a bond $\{u, v\}$ to be *occupied* with probability $pD(v-u)$ and *vacant* with probability $1-pD(v-u)$. The occupation status of all bonds are independent random variables. The law of the configuration of occupied bonds (at the critical percolation threshold) is denoted by $\mathbb{P}_{p_c}$ with corresponding expectation denoted by $\mathbb{E}_{p_c}$. Given a configuration we say that x is connected to y, and write $x \xrightarrow{n} y$, if there is a path of occupied bonds from x to y, and the path with minimal number of bonds connecting x and y has precisely n edges. For percolation, Condition 1.1 is not known. The bound $\theta_n \leq C/n$, which can be used instead of Condition 1.3, is proved in [21] (in fact, we use Condition 1.3 together with an adaptation of the argument in [21] precisely to prove that $\theta_n \leq C/n$ in our general setting). Condition 1.2 follows from [7] together with [2], see also [8, 9]. As a result, for percolation, our results hold as soon as Condition 1.1 is proved.

Our main result can be restated in terms of the above models as follows.

Theorem 1.5. *Let $L \gg 1$, and let $d > 4$ for spread-out oriented percolation and the spread-out contact process, and $d > 8$ for spread-out lattice trees. Then, with $A, V, v > 0$ all depending on L such that*

$$\frac{1}{A(VA^2n)^{r-2}} \hat{t}^{(r)}_{n\vec{t}}\Big(\vec{k}/\sqrt{vn}\Big) \to \widehat{M}^{(r-1)}_{\vec{t}}(\vec{k}), \quad \text{as } n \to \infty, \tag{1.17}$$

the asymptotics

$$n\theta_n \to 2/(AV) \quad \text{and} \quad \mu_n(X^{(n)}_t(1) > 0) \to \mathbb{N}_0(X_t(1) > 0) = 2/t, \quad \text{as } n \to \infty \tag{1.18}$$

hold. As a consequence, the finite-dimensional distributions of the process $(X^{(n)}_t)_{t>0}$ under μ_n converge to those of $(X_t)_{t>0}$ under the measure $\mathbb{N}_0$.

The remainder of this paper is organized as follows. In Section 2, we prove an upper bound on θ_n that is of the correct order, but with the wrong constant. In Section 3, we use weak-convergence arguments to identify the correct constant.

2 Weak upper bound on the survival probability

The following theorem gives a weak upper bound on the survival probability.

Theorem 2.1. *When (1.9) and (1.10) hold, there exists a constant c_+ such that*

$$\theta_n \leq c_+/n. \tag{2.1}$$

Proof. We follow [21], where a similar bound was proved for the intrinsic one-arm in percolation. We split θ_{4n} into two parts,

$$\theta_{4n} = \mathbb{P}(N_m \geq \varepsilon n\ \forall m \in [n, 3n], 0 \longrightarrow 4n) + \mathbb{P}(\exists m \in [n, 3n] \colon N_m < \varepsilon n, 0 \longrightarrow 4n). \tag{2.2}$$

We can bound the first probability using (1.9), since if $N_m \geq \varepsilon n$ for all $m \in [n, 3n]$, we have that $|\mathcal{C}(0)| \geq 2\varepsilon n^2$. Therefore,

$$\mathbb{P}(N_m \geq \varepsilon n\ \forall m \in [n, 3n], 0 \longrightarrow 4n) \leq \mathbb{P}(|\mathcal{C}(0)| \geq 2\varepsilon n^2) \leq \frac{C_{\mathcal{C}}}{n\sqrt{2\varepsilon}}. \tag{2.3}$$

In the second probability in (2.2), we let $J \geq n$ be the first $m \in [n, 3n]$ such that $0 < N_m < \varepsilon n$, and we condition on $\mathcal{F}_J = \sigma((A_m)_{m \leq J})$. Then, by (1.10),

$$\mathbb{P}(A_J \longrightarrow 4n \mid \mathcal{F}_J) \leq N_J C_\theta \theta_n \leq \varepsilon n C_\theta \theta_n. \tag{2.4}$$

As a result,

$$\mathbb{P}(\exists m \in [n, 3n]\colon N_m < \varepsilon n, 0 \longrightarrow 4n) = \mathbb{E}[\mathbb{1}_{\{n \le J \le 3n\}} \mathbb{P}(A_J \longrightarrow 4n \mid \mathcal{F}_J)] \le \varepsilon C_\theta n \theta_n^2, \tag{2.5}$$

where we use the fact that $n \le J$ implies that $0 \longrightarrow n$. Thus, we end up with the inequality

$$\theta_{4n} \le \frac{C_c}{n\sqrt{2\varepsilon}} + \varepsilon C_\theta n \theta_n^2. \tag{2.6}$$

Take $\varepsilon = c_2^{-4/3}$ and take $c_2 > 1$ so large that

$$2^{-\frac{1}{2}} C_c c_2^{2/3} + C_\theta c_2^{2/3} \le c_2/4. \tag{2.7}$$

Then, it is easy to prove by induction that $\theta_{4^k} \le c_2 4^{-k}$ for every $k \ge 1$. By monotonicity of $n \mapsto \theta_n$, this immediately implies that $\theta_n \le (4c_2)/n$. This completes the proof of Theorem 2.1. ∎

3 Identifying the constant: Proof of Theorem 1.4

In this section, we make use of general weak convergence arguments to prove that $n\theta_n \to 2/(AV)$. We rely crucially on [20, Proposition 2.3] (restated below as Proposition 3.1), which requires the introduction of some more notation. Let $M_F(\mathbb{R}^d)$ denote the space of finite measures on $\mathbb{R}^d$ equipped with the topology of weak convergence, and let $M_F(\mathbb{R}^d)^m$ denote the corresponding product space with product topology. A function $Q\colon M_F(\mathbb{R}^d)^m \to \mathbb{R}$ is said to be a multinomial if $Q(\vec{X})$ is a real multinomial in $\{X_1(1), \ldots, X_m(1)\}$. Let $\mathcal{D}_F$ denote the set of discontinuities of a function F, and $D(E)$ denote the space of càdlàg E-valued functions with the Skorohod topology. When we say that μ is a measure on (a topological space) E, this means that it is a measure with respect to the Borel σ-algebra on E.

Proposition 3.1. *Let $\{\mu_n\}_{n\ge1}$ be a sequence of finite measures on $D(M_F(\mathbb{R}^d))$ such that Condition 1.1 holds (for each $r, \vec{k}, \vec{t}$). Then, for every $s > 0$, $\eta > 0$, $m \ge 1$, $\vec{t} \in [0,\infty)^m$ and every Borel measurable $F\colon M_F(\mathbb{R}^d)^m \to \mathbb{C}$ bounded by a multinomial and such that $\mathbb{N}_0\big((X_{t_1}, \ldots, X_{t_m}) \in \mathcal{D}_F\big) = 0$,*

(i)
$$\mathbb{E}_{\mu_n}\left[X_s^{(n)}(1) F(\vec{X}_{\vec{t}}^{(n)})\right] \to \mathbb{E}_{\mathbb{N}_0}\left[X_s(1) F(\vec{X}_{\vec{t}})\right], \tag{3.1}$$

and

(ii)
$$\mathbb{E}_{\mu_n}\left[F(\vec{X}_{\vec{t}}^{(n)}) \mathbb{1}_{\{X_s^{(n)}(1) > \eta\}}\right] \to \mathbb{E}_{\mathbb{N}_0}\left[F(\vec{X}_{\vec{t}}) \mathbb{1}_{\{X_s(1) > \eta\}}\right]. \tag{3.2}$$

Proof of Theorem 1.4. By Theorem 2.1, we have that $n\theta_n$ is bounded. In order to investigate the limit of $n\theta_n$, we split, for each fixed $\varepsilon > 0$,

$$n\theta_n \quad = \quad n\mathbb{P}(N_n > \varepsilon n) + n\mathbb{P}(0 < N_n \le \varepsilon n). \tag{3.3}$$

The first term is equal to $(AV)^{-1}\mu_n(X_1^{(n)} > c\varepsilon)$, with $c = (VA^2)^{-1}$. From Proposition 3.1 with $s = 1$, $\eta = c\varepsilon$ and with the continuous function $F \equiv 1$ (and Condition 1.1), we have that the first term on the right converges to $(AV)^{-1}\mathbb{N}_0(X_1(1) > c\varepsilon)$, and this converges to $(AV)^{-1}\mathbb{N}_0(X_1(1) > 0) = 2/(AV)$ as $\varepsilon \to 0$. Since $n\mathbb{P}(0 < N_n \le \varepsilon n) \ge 0$, this immediately proves that

$$\liminf_{n\to\infty} n\theta_n \ge 2/(AV). \tag{3.4}$$

In order to identify the limit, we proceed as in [11]. Let $\delta \in (0,1)$ and let $\{n_k\} = \{n_k(\delta)\}$ be any subsequence of $\mathbb{N}$ such that $n_k\theta_{n_k} \to \limsup_n n\theta_n = \bar{b}$, and $(1-\delta)n_k\theta_{(1-\delta)n_k} \to b_\delta$ for some $b_\delta \geq 2/AV$. This can be achieved by first taking a subsequence $\{m_l\}$ for which $m_l\theta_{m_l} \to \bar{b}$, and then taking a further subsequence $\{m_{l_k}\}$ such that $(1-\delta)m_{l_k}\theta_{(1-\delta)m_{l_k}} \to b_\delta$. The required sequence is then $n_k = m_{l_k}$.

Similarly to (3.3), for $\delta, \varepsilon, \varepsilon' \in (0,1)$ we write

$$\begin{aligned} n_k\theta_{n_k} &= n_k\mathbb{P}(N_{(1-\delta)n_k} > \varepsilon n_k, N_{n_k} > \varepsilon' n_k) \\ &\quad + n_k\mathbb{P}(N_{(1-\delta)n_k} > \varepsilon n_k, 0 < N_{n_k} \leq \varepsilon' n_k) + n_k\mathbb{P}(0 < N_{(1-\delta)n_k} \leq \varepsilon n_k, N_{n_k} > 0) \\ &= A_{k,\delta,\varepsilon,\varepsilon'} + B_{k,\delta,\varepsilon,\varepsilon'} + D_{k,\delta,\varepsilon}. \end{aligned} \tag{3.5}$$

In particular then, for each $\delta, \varepsilon, \varepsilon'$

$$\bar{b} = \limsup_{k\to\infty} n_k\theta_{n_k} \leq \limsup_{k\to\infty} A_{k,\delta,\varepsilon,\varepsilon'} + \limsup_{k\to\infty} B_{k,\delta,\varepsilon,\varepsilon'} + \limsup_{k\to\infty} D_{k,\delta,\varepsilon}. \tag{3.6}$$

It follows then that also

$$\bar{b} \leq \limsup_{\delta,\varepsilon,\varepsilon'\downarrow 0} \limsup_{k\to\infty} A_{k,\delta,\varepsilon,\varepsilon'} + \limsup_{\delta,\varepsilon,\varepsilon'\downarrow 0} \limsup_{k\to\infty} B_{k,\delta,\varepsilon,\varepsilon'} + \limsup_{\delta,\varepsilon\downarrow 0} \limsup_{k\to\infty} D_{k,\delta,\varepsilon}, \tag{3.7}$$

where the limits are taken in the order $k \to \infty$, $\varepsilon' \downarrow 0$, $\varepsilon \downarrow 0$, $\delta \downarrow 0$.

The term $A_{k,\delta,\varepsilon,\varepsilon'}$ can be rewritten as

$$\frac{1}{AV}\mu_{n_k}(X^{(n_k)}_{1-\delta}(1) > c\varepsilon, X^{(n_k)}_1(1) > c\varepsilon') \to \frac{1}{AV}\mathbb{N}_0(X_{1-\delta}(1) > c\varepsilon, X_1(1) > c\varepsilon'), \quad \text{as } k \to \infty,$$

by (3.2). Letting $\varepsilon' \downarrow 0$ and then $\varepsilon \downarrow 0$ this converges to

$$\frac{1}{AV}\mathbb{N}_0(X_{1-\delta}(1) > 0, X_1(1) > 0) = \frac{1}{AV}\mathbb{N}_0(X_1(1) > 0) = 2/AV,$$

which, in particular, does not depend on δ.

It therefore remains to show that

$$\limsup_{\delta,\varepsilon,\varepsilon'\downarrow 0} \limsup_{k\to\infty} B_{k,\delta,\varepsilon,\varepsilon'} + \limsup_{\delta,\varepsilon\downarrow 0} \limsup_{k\to\infty} D_{k,\delta,\varepsilon} = 0.$$

Using Condition 1.3, the term $D_{k,\delta,\varepsilon}$ satisfies

$$D_{k,\delta,\varepsilon} = n_k\mathbb{E}\big[I_{\{0<N_{(1-\delta)n_k}\leq\varepsilon n_k\}}\mathbb{P}(N_{n_k} > 0|\mathcal{F}_{(1-\delta)n_k})\big] \leq C_\theta\varepsilon n_k\theta_{\delta n_k} n_k\theta_{(1-\delta)n_k} \leq \frac{C\varepsilon}{\delta(1-\delta)},$$

uniformly in k, where we have used the fact that $n\theta_n$ is bounded above uniformly in k. Letting $\varepsilon \downarrow 0$, this converges to 0.

It therefore remains to show that

$$\limsup_{\delta,\varepsilon,\varepsilon'\downarrow 0} \limsup_{k\to\infty} B_{k,\delta,\varepsilon,\varepsilon'} = 0. \tag{3.8}$$

For each m, define the measure $\mathbb{Q}_m = \mathbb{P}(\cdot \mid N_m > 0)$. Then,

$$B_{k,\delta,\varepsilon,\varepsilon'} = n_k\theta_{(1-\delta)n_k}\mathbb{Q}_{(1-\delta)n_k}(N_{(1-\delta)n_k} > \varepsilon n_k, 0 < N_{n_k} \leq \varepsilon' n_k).$$

Thus, since $n_k\theta_{(1-\delta)n_k}$ is bounded above by $\frac{C}{1-\delta} \leq 2C$ for $\delta < \frac{1}{2}$ (where C is independent of δ), proving (3.8) is equivalent to proving that

$$\limsup_{\delta,\varepsilon,\varepsilon'\downarrow 0} \limsup_{k\to\infty} \mathbb{Q}_{(1-\delta)n_k}(N_{(1-\delta)n_k} > \varepsilon n_k, 0 < N_{n_k} \leq \varepsilon' n_k) = 0.$$

We note that, for any integers $\ell_1, \ell_2 \geq 0$ such that $\ell_1 + \ell_2 \geq 1$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_{(1-\delta)n_k}}\Big[\Big(N_{(1-\delta)n_k}/n_k\Big)^{\ell_1}\Big(N_{n_k}/n_k\Big)^{\ell_2}\Big] &= \frac{1}{\theta_{(1-\delta)n_k}}\mathbb{E}\Big[\Big(N_{(1-\delta)n_k}/n_k\Big)^{\ell_1}\Big(N_{n_k}/n_k\Big)^{\ell_2}\Big] \\ &= \frac{1}{n_k\theta_{(1-\delta)n_k}} n_k^{-(\ell_1+\ell_2-1)}\mathbb{E}[N^{\ell_1}_{(1-\delta)n_k}N^{\ell_2}_{n_k}] \\ &= \frac{1}{n_k\theta_{(1-\delta)n_k}} n_k^{-(\ell_1+\ell_2-1)}\hat{t}^{(\ell_1+\ell_2+1)}_{\vec{n}_k}(0),\end{aligned} \tag{3.9}$$

where we use that $N_{(1-\delta)n_k} > 0$ when $N_{n_k} > 0$, and where $\vec{n}_k$ denotes a vector with precisely ℓ_1 coordinates equal to $(1-\delta)n_k$ and ℓ_2 coordinates equal to n_k. By Proposition 3.1,

$$\begin{aligned}n_k^{-(\ell_1+\ell_2-1)}\hat{\tau}^{(\ell_1+\ell_2+1)}_{\vec{n}_k}(0) &\to A(VA^2)^{\ell_1+\ell_2-1}\mathbb{E}_{\mathbb{N}_0}\Big[X_{1-\delta}(1)^{\ell_1}X_1(1)^{\ell_2}\Big] \\ &= \frac{2}{AV(1-\delta)}\mathbb{E}_{\mathbb{N}_0}\Big[\Big(VA^2X_{1-\delta}(1)\Big)^{\ell_1}\Big(VA^2X_1(1)\Big)^{\ell_2}\Big|X_{1-\delta}(1)>0\Big],\end{aligned} \tag{3.10}$$

where the last equality follows from the fact that $\mathbb{N}_0(X_{1-\delta}(1) > 0) = 2/(1-\delta)$. Therefore, also using that $(1-\delta)n_k\theta_{(1-\delta)n_k} \to b_\delta$,

$$\mathbb{E}_{\mathbb{Q}_{(1-\delta)n_k}}\Big[\Big(N_{(1-\delta)n_k}/n_k\Big)^{\ell_1}\Big(N_{n_k}/n_k\Big)^{\ell_2}\Big] \to \frac{2}{AVb_\delta}\mathbb{E}_{\mathbb{N}_0}\Big[\Big(VA^2X_{1-\delta}(1)\Big)^{\ell_1}\Big(VA^2X_1(1)\Big)^{\ell_2}\Big|X_{1-\delta}(1)>0\Big].$$

We recognize the above joint moments as the joint moments of (X, Y) with distribution $(1-\alpha_\delta)\delta_{(0,0)}+\alpha_\delta\nu_\delta$, where $\delta_{(0,0)}$ is the point measure on the vector $(0, 0)$ and ν_δ is the law of $(A^2VX_{1-\delta}(1), A^2VX_1(1))$ under $\mathbb{N}_0(\cdot|X_{1-\delta}(1) > 0)$, and with $\alpha_\delta = 2/(AVb_\delta)$. In particular, for any $t > 1-\delta$,

$$\nu_\delta(X_t(1) = 0) = 1 - (1-\delta)/t, \tag{3.11}$$

so that

$$\nu_\delta(X_1(1) = 0) = 1 - (1-\delta) = \delta. \tag{3.12}$$

Let (X_n, Y_n) be a two dimensional distribution. By [10, Theorem I], convergence of the joint moments of (X_n, Y_n) to those of (X, Y) implies convergence in distribution when the joint moments identify the distribution. Let

$$\lambda_k = \mathbb{E}[X^k + Y^k]. \tag{3.13}$$

By [29, Lemma 1.4(3)] and the fact that $X, Y \geq 0$, the joint moments of (X, Y) determine their distribution when

$$\sum_{k\geq 1}\lambda_k^{-1/(2k)} = +\infty. \tag{3.14}$$

Under the conditional law $\mathbb{N}_0(\cdot|X_{1-\delta}(1) > 0)$, the distribution of $A^2VX_{1-\delta}(1)$ is exponential with mean $(1-\delta)A^2V/2$ (see e.g., [11, Theorem 1.4]), and by (3.12), $A^2VX_1(1)$ is 0 with probability δ and an exponential with mean $A^2V/2$ with probability $1-\delta$.

As a result, the distribution of both limits X and Y are mixtures of point masses at 0 with probabilities $1-\alpha_\delta$ and $1-\alpha_\delta+\alpha_\delta\delta$ and exponentials with positive means λ_X and λ_Y. Therefore, for all $k \geq 0$,

$$\lambda_k = \Big(\alpha_\delta\lambda_X^k + \alpha_\delta(1-\delta)\lambda_Y^k\Big)k!, \tag{3.15}$$

which implies that (3.14) holds, so that $\big(N_{(1-\delta)n_k}/n_k, N_{n_k}/n_k\big)$ converges in distribution to (X, Y) having distribution $(1-\alpha_\delta)\delta_{(0,0)} + \alpha_\delta\nu_\delta$.

Thus, as $k \to \infty$, we have that

$$\mathbb{Q}_{(1-\delta)n_k}(N_{(1-\delta)n_k} > \varepsilon n_k, N_{n_k} \leq \varepsilon' n_k) \to \alpha_\delta\nu_\delta(A^2VX_{1-\delta}(1) > \varepsilon, A^2VX_1(1) \leq \varepsilon').$$

When $\varepsilon' \downarrow 0$, we obtain that

$$\nu_\delta(A^2VX_{1-\delta}(1) > \varepsilon, A^2VX_1(1) \leq \varepsilon') \to \nu_\delta(X_{1-\delta}(1) > \varepsilon A^2V, X_1(1) = 0) \leq \nu_\delta(X_1(1) = 0) = \delta, \tag{3.16}$$

where we use (3.12). Letting $\delta \downarrow 0$, we obtain the desired result. ■

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