STATS 325
STATS 793

THE UNIVERSITY OF AUCKLAND

MOCK EXAMINATION
Campus: City

STATISTICS

Stochastic Processes

(Time allowed: THREE hours)

NOTE: Attempt **ALL** questions. Marks for each question are shown in brackets. An Attachment containing useful information is found on page 4.

1. Let A and B be any events.

   (a) Show that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(\overline{A} \cap B)$. (3 M)

   (b) Starting from the result in part (a), show that $\mathbb{P}(A \cup B) = 1 - \mathbb{P}(\overline{A} \cap \overline{B})$. (4 M)

2. Tom is trying to find his lost golf ball. He walks along a line looking to left and right. Suppose that the golf ball is distance X away from the line, where X is a random variable measured perpendicular to the line, with distribution $X \sim \text{Uniform}(0, 5)$ metres. If the golf ball is distance x away from the line, Tom will see it with probability e^{-x}.

   Find the overall probability that Tom finds his lost ball. (5 M)

3. (a) Let X be a discrete random variable with probability generating function $G_X(s)$. Let $Y = aX + b$, where a and b are constants. Let $G_Y(s)$ be the probability generating function of Y. Show that

   $$G_Y(s) = s^b G_X(s^a).$$

   (3 E)

   (b) Let $X \sim \text{Binomial}(5, 0.4)$. Show that

   $$G_X(s) = (0.4s + 0.6)^5,$$

   and state the radius of convergence. (4 E)

   (c) Let $Y = 2X + 1$, where $X \sim \text{Binomial}(5, 0.4)$ as above. Find $G_Y(s)$. (1 E)

4. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and suppose that $Y \sim \text{Binomial}(2, \frac{3}{4})$.

   (a) Let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. Using the information in the Attachment, state $G(s)$. (1 E)

   (b) Let $G_2(s)$ be the probability generating function of Z_2. Find $G_2(s)$. (Do **not** simplify the expression.) (2 E)

   (c) Show that $\mathbb{P}(Z_4 = 0) = 0.106$. (3 M)

   (d) Find the probability of eventual extinction, γ. (3 E)

   (e) Suppose that $Z_6 = 8$. Find the probability of eventual extinction. (2 E)

   (f) Suppose again that $Z_6 = 8$. Find the probability that exactly 5 of the 8 individuals that were alive at generation 6 still have living descendents in generation 10.
   [Hint: use part (c).] (4 M)

5. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. You may assume that $G(s)$ is strictly convex for $0 \le s \le 1$. Let $\mu = \mathbb{E}(Y)$, and let γ be the probability of eventual extinction.

   Using diagrams, show that $\gamma = 1$ if $\mu < 1$, and $\gamma < 1$ if $\mu > 1$. You must clearly label all parts of the diagram, and state any theorems that you use. (8 M)

6. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3\}$, with transition matrix

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}.$$

   (a) Suppose that $X_0 \sim 1 + \text{Binomial}(2, 0.4)$. Find a vector describing the distribution of X_1. (5 E)

   (b) Find $\mathbb{P}(X_1 = 1, X_2 = 2, X_3 = 3 \mid X_0 = 2)$. (2 E)

   (c) **NOT EXAMINABLE FROM 2006 ONWARDS.**
   (Find a general formula for $\mathbb{P}(X_t = 3 \mid X_0 = 1)$, for any t.) (14 M)

7. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3\}$, with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}.$$

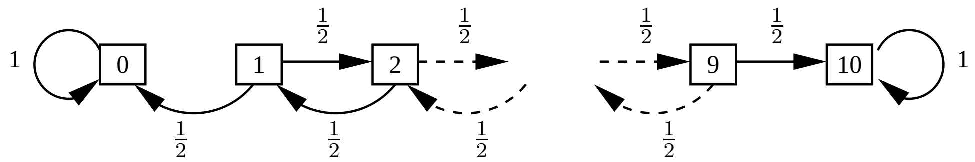
   (a) Draw the transition diagram. (2 E)

   (b) Find an equilibrium distribution for P. (4 E)

   (c) Does X_t converge to the distribution in (b) as $t \to \infty$? Explain why or why not. (3 M)

8. A gambler starts with \$1. He tosses a fair coin repeatedly. If he gets a Head, he wins \$1. If he gets a Tail, he loses \$1.

   The gambler will stop when he has either \$0 or \$10. Let $\{X_0, X_1, \ldots\}$ denote a Markov chain denoting the gambler's money after toss t, where $X_0 = \$1$. The transition diagram is below.



   (a) Define h_k to be the probability that the gambler eventually wins \$10, starting from \$k. State the values of h_0 and h_{10}, and prove by induction that

$$h_k = \left(\frac{k}{k+1}\right) h_{k+1} \quad \text{for } k = 1, 2, \ldots, 9.$$

(10 H)

   (b) Deduce the probability that the gambler wins \$10, starting from \$1. (3 M)

   (c) Let T_1 be the time taken for the gambler to reach \$10, starting from \$1. State $\mathbb{E}(T_1)$. (1 M)

9. Let $\{X_1, X_2, X_3, \ldots\}$ be a Markov chain where X_t represents the *maximum* score obtained after t tosses of a fair 6-sided dice. (Thus each X_t can take values 1, 2, 3, 4, 5, 6.) Find the transition matrix of the Markov chain. (8 H)

10. Suppose there are 3 levels in a building: floors 1, 2, and 3. A lift is available to travel between the floors. People travel between the floors according to the following transition matrix:

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{7}{8} & 0 & \frac{1}{8} \\ \frac{7}{8} & \frac{1}{8} & 0 \end{pmatrix}.$$

   This means that a person on floor 1 travels to floor 2 with probability $\frac{1}{3}$, and to floor 3 with probability $\frac{2}{3}$, and so on.

   The lift can be summoned from any floor: people press the 'up' button when they wish to travel upwards, and the 'down' button when they wish to travel downwards. The lift always behaves sensibly: it will stop to let people on only when it is travelling in the same direction as they wish to go. If people on different floors wish to travel in different directions, the lift gives precedence to the people who get in first.

   Suppose that the lift is at floor 1, and there is one person on floor 1 and one person on floor 2 who wish to take the lift. Let X be the floor number of the lift's next stop. Find the probability distribution of X. (5 M)

ATTACHMENT

1. **Discrete Probability Distributions**

Distribution	$\mathbb{P}(X = x)$	$\mathbb{E}(X)$	PGF, $\mathbb{E}(s^X)$
Geometric(p)	pq^x (where $q = 1 - p$), for $x = 0, 1, 2, \ldots$	$\dfrac{q}{p}$	$\dfrac{p}{1 - qs}$
	Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Binomial(n, p)	$\binom{n}{x} p^x q^{n-x}$ (where $q = 1 - p$) for $x = 0, 1, 2, \ldots, n$.	np	$(ps + q)^n$
	Number of successes in n independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Poisson(λ)	$\dfrac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \ldots$	λ	$e^{\lambda(s-1)}$

2. **Uniform Distribution:** $X \sim \text{Uniform}(a, b)$.
Probability density function, $f_X(x) = \frac{1}{b-a}$ for $a < x < b$. Mean, $\mathbb{E}(X) = \frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

Definition:	$G_X(s) = \mathbb{E}(s^X)$	
Moments:	$\mathbb{E}(X) = G_X'(1)$	$\mathbb{E}\left\{X(X-1)\ldots(X-k+1)\right\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \dfrac{1}{n!} G_X^{(n)}(0)$	

4. **Geometric Series:** $1 + r + r^2 + r^3 + \ldots = \displaystyle\sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$ for $|r| < 1$.

Finite sum: $\displaystyle\sum_{x=0}^{n} r^x = \frac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p, q \in \mathbb{R}$, and integer n, $(p + q)^n = \displaystyle\sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\displaystyle\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$.