Stats 325 Mock 1 Solutions

1) a) $$\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A\cap B) \quad \circledast$$
$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A|B)\mathbb{P}(B)$$
$$= \mathbb{P}(A) + \mathbb{P}(B)(1-\mathbb{P}(A|B))$$
$$= \mathbb{P}(A) + \mathbb{P}(B)\mathbb{P}(\bar{A}|B)$$
$$\underline{\underline{\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(\bar{A}\cap B)}} \quad \text{as required.}$$

Alternative: note that $\mathbb{P}(B) = \mathbb{P}(A\cap B) + \mathbb{P}(\bar{A}\cap B)$ by Partition Rule,
so $\mathbb{P}(B) - \mathbb{P}(A\cap B) = \mathbb{P}(\bar{A}\cap B)$.

Thus by $\circledast$, $\underline{\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(\bar{A}\cap B)}$.

b) $\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(\bar{A}\cap B)$ from (a). $\circledast\circledast$

By the Partition Rule, $\mathbb{P}(\bar{A}\cap\bar{B}) + \mathbb{P}(\bar{A}\cap B) = \mathbb{P}(\bar{A})$.

So $\mathbb{P}(\bar{A}\cap B) = \mathbb{P}(\bar{A}) - \mathbb{P}(\bar{A}\cap\bar{B})$.

Substituting into $\circledast\circledast$: $\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(\bar{A}) - \mathbb{P}(\bar{A}\cap\bar{B})$

$\therefore$ $\underline{\underline{\mathbb{P}(A\cup B) = 1 - \mathbb{P}(\bar{A}\cap\bar{B})}}$ as stated.

2) $X \sim \text{Uniform}(0,5)$, so p.d.f. of X is $f_X(x) = \frac{1}{5}$ $(0<x<5)$.

Given information: $\mathbb{P}(\text{see ball} \mid \text{ball is at } x) = e^{-x}$.

Thus $$\mathbb{P}(\text{see ball}) = \mathbb{E}_X\{\mathbb{P}(\text{see ball} \mid X)\}$$ (Probability as a Conditional Expectation: Section 3.8)
$$= \int_{x=0}^{5} \mathbb{P}(\text{see ball} \mid X=x)\, f_X(x)\, dx$$
$$= \int_{x=0}^{5} e^{-x} \cdot \frac{1}{5}\, dx$$

2 cont.) $$\mathbb{P}(\text{see ball}) = \frac{1}{5}\left[-e^{-x}\right]_0^5$$
$$\underline{\underline{\mathbb{P}(\text{see ball}) = \frac{1}{5}(1-e^{-5})}} \quad (0.199).$$

3a) $$G_Y(s) = \mathbb{E}(s^Y)$$
$$= \mathbb{E}(s^{aX+b}) \quad \text{as } Y = aX+b$$
$$= \mathbb{E}(s^{aX}s^b)$$
$$= s^b\,\mathbb{E}(s^{aX}) \quad (s^b \text{ is a constant})$$
$$= s^b\,\mathbb{E}((s^a)^X)$$
$$\underline{\underline{G_Y(s) = s^b G_X(s^a)}} \quad \text{as stated.}$$

b) $X \sim \text{Binomial}(5, 0.4)$, so $\mathbb{P}(X=x) = \binom{5}{x} 0.4^x\, 0.6^{5-x}$ for $x = 0, \dots, 5$.

Then:
$$G_X(s) = \mathbb{E}(s^X) = \sum_{x=0}^{5} s^x\,\mathbb{P}(X=x)$$
$$= \sum_{x=0}^{5}\binom{5}{x}(0.4)^x(0.6)^{5-x}\cdot s^x$$
$$= \sum_{x=0}^{5}\binom{5}{x}(0.4s)^x(0.6)^{5-x}$$
$$= \underline{\underline{(0.4s+0.6)^5}} \quad \text{by the Binomial Theorem (see Attachment).}$$

The working is valid for all $s\in\mathbb{R}$, so $\underline{\underline{\text{radius of convergence} = \infty}}$.

c) $Y = 2X+1 \Rightarrow G_Y(s) = s^1 G_X(s^2)$ by part (a): $a=2$, $b=1$.

So
$$\underline{\underline{G_Y(s) = s\{0.4s^2 + 0.6\}^5 \quad \text{for } s\in\mathbb{R}.}}$$

4) a) $Y \sim \text{Binomial}\left(2, \frac{3}{4}\right)$, so $\underline{\underline{G(s) = \left(\frac{3}{4}s + \frac{1}{4}\right)^2}}$.

b) $G_2(s) = G(G(s))$

$= \left(\frac{3}{4}G(s) + \frac{1}{4}\right)^2$

$\Rightarrow \underline{\underline{G_2(s) = \left\{ \frac{3}{4}\left(\frac{3}{4}s + \frac{1}{4}\right)^2 + \frac{1}{4} \right\}^2}}$. (*)

c) $\mathbb{P}(Z_4 = 0) = G_4(0)$

$= G_2(G_2(0))$ (or $G(G(G(G(0))))$)

$= G_2(0.088)$ by substitution in (*)

$\underline{\underline{= 0.106}}$ as stated.

d) γ is the minimal non-negative solution to the equation

$G(s) = s$

$\Rightarrow \left(\frac{3}{4}s + \frac{1}{4}\right)^2 = s$

$\Rightarrow \frac{9}{16}s^2 + \frac{6}{16}s + \frac{1}{16} = s$

$\Rightarrow 9s^2 - 10s + 1 = 0$

$(s-1)(9s-1) = 0$ (know $s=1$ is a solution)

$\Rightarrow \underline{s=1, \quad s = 1/9}$.

So γ is the smaller solution, $\Rightarrow$ $\underline{\underline{\mathbb{P}(\text{extinction}) = \gamma = \frac{1}{9}}}$.

e) If $Z_6 = 8$, we have 8 independent branching processes beginning at time 6.

$\mathbb{P}(\text{extinction}) = \mathbb{P}(\text{all 8 become extinct}) = \gamma^8 = \left(\frac{1}{9}\right)^8 = 2.3 \times 10^{-8}$.

4f) We have 8 independent branching processes, begining in generation 6.

For each process, the #individuals in generation 10 has the same distribution as Z_4 (starting from 1 individual at $t=0$).

So $\mathbb{P}(\text{descendents still alive at } t=10) = 1 - \mathbb{P}(Z_4 = 0)$

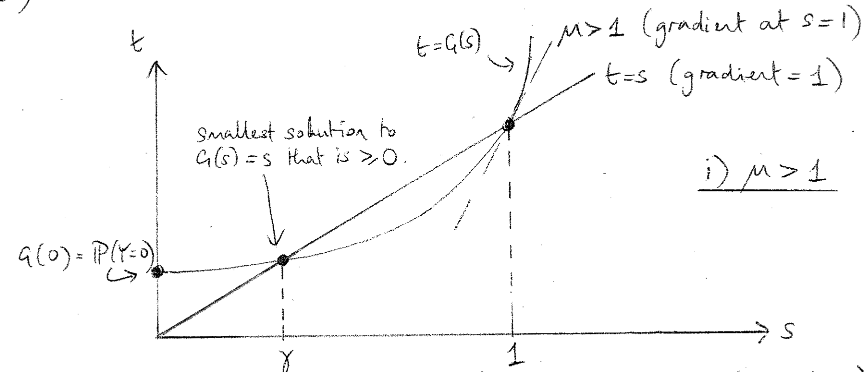
$= 1 - 0.106$

$= \underline{0.894}$ for each strain.

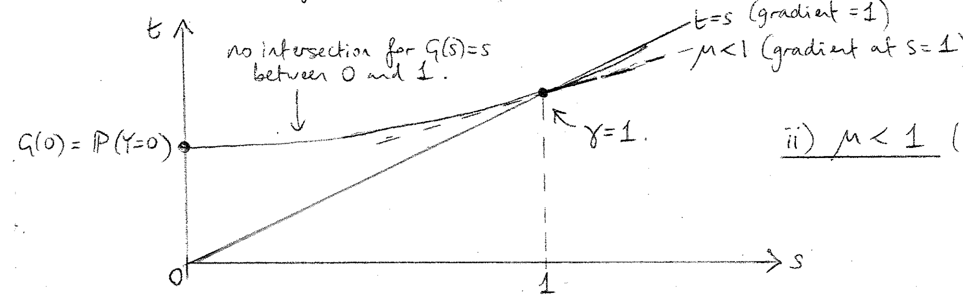
Thus $\mathbb{P}(5 \text{ out of } 8 \text{ still alive}) = \binom{8}{5}(0.894)^5(0.106)^3$

$\underline{\underline{= 0.038}}$.

5)



i) $\underline{\mu > 1}$ $(\gamma < 1)$



ii) $\underline{\mu < 1}$ $(\gamma = 1)$

γ is the minimal solution ≥ 0 to the equation $G(s) = s$.
μ is the gradient of $G(s)$ at $s=1$ $(\mu = G'(1))$.
The graph $G(s)$ cuts the t-axis at $\mathbb{P}(Y=0)$, which is ≥ 0.
Thus, when $\mu > 1$, the curve $G(s)$ is forced below the line $t=s$ at $s=1$, so (by convex shape) must cross the line $t=s$ again to meet the t-axis, giving an intersection at $\gamma < 1$. When $\mu < 1$, no such intersection is possible.

(5)

6.) a) $X_0 \sim 1 + \text{Bin}(2, 0.4)$ (sample space is $\{1,2,3\}$ not $\{0,1,2\}$)

$$\text{So} \quad \mathbb{P}(X_0 = 1) = \binom{2}{0}(0.4)^0(0.6)^2 = 0.36$$
$$\mathbb{P}(X_0 = 2) = \binom{2}{1}(0.4)(0.6) = 0.48$$
$$\mathbb{P}(X_0 = 3) = \binom{2}{2}(0.4)^2(0.6)^0 = 0.16$$

[uses Bin. formula for 0, 1, 2]

$$\text{Thus} \quad X_0 \sim (0.36,\ 0.48,\ 0.16)^T$$

$$\text{So} \quad X_1 \sim (0.36,\ 0.48,\ 0.16)^T\ P$$

$$= \begin{pmatrix} 0.36 & 0.48 & 0.16 \end{pmatrix} \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}$$

$$\underline{\underline{X_1 \sim (0.408,\ 0.2,\ 0.392)}}$$

b) $\mathbb{P}(X_1 = 1,\ X_2 = 2,\ X_3 = 3 \mid X_0 = 2)$

$$= \mathbb{P}(X_1 = 1 \mid X_0 = 2)\ \mathbb{P}(X_2 = 2 \mid X_1 = 1)\ \mathbb{P}(X_3 = 3 \mid X_2 = 2)$$

$$= p_{21} \cdot p_{12} \cdot p_{23}$$

$$= 0.4 * 0.2 * 0.6$$

$$\underline{\underline{= 0.048}}.$$

c) $\mathbb{P}(X_t = 3 \mid X_0 = 1) = (P^t)_{13}$

Now if P has distinct eigenvalues $\lambda_1 = 1,\ \lambda_2,\ \lambda_3$, then we can find a general solution

$$(P^t)_{13} = c_1 + c_2 \lambda_2^t + c_3 \lambda_3^t,$$

where constants c_1, c_2, c_3 are to be determined.

(6)

6c cont.) Find the eigenvalues of P:

$$\det(P - \lambda I) = \begin{vmatrix} 0.6-\lambda & 0.2 & 0.2 \\ 0.4 & -\lambda & 0.6 \\ 0 & 0.8 & 0.2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (0.6-\lambda)\{-\lambda(0.2-\lambda) - 0.48\}$$
$$- 0.2\,(0.4(0.2-\lambda)) + 0.2\,(0.32) = 0$$

$$-\lambda(0.2-\lambda)(0.6-\lambda) - 0.48(0.6-\lambda)$$
$$- 0.016 + 0.08\lambda + 0.064 = 0$$

$$-\lambda(0.12 - 0.8\lambda + \lambda^2) - 0.24 + 0.56\lambda = 0$$

$$-\lambda^3 + 0.8\lambda^2 + 0.44\lambda - 0.24 = 0$$

$$\Rightarrow \quad \lambda^3 - 0.8\lambda^2 - 0.44\lambda + 0.24 = 0$$

$$(\lambda - 1)(\lambda^2 + 0.2\lambda - 0.24) = 0$$

must be a factor

$$\lambda = 1, \quad \text{or} \quad \lambda = \frac{-0.2 \pm \sqrt{(0.2)^2 + 4(0.24)}}{2}$$

$$\Rightarrow \quad \lambda = 0.4, \quad \lambda = -0.6.$$

Thus the matrix has distinct eigenvalues, $\underline{\underline{\lambda_1 = 1,\ \lambda_2 = 0.4,\ \lambda_3 = -0.6.}}$

General solution:

$$\underline{(P^t)_{13} = c_1 + c_2(0.4)^t + c_3(-0.6)^t}$$

Initial conditions:

(t=0) $\quad (P^0)_{13} = 0 \quad \Rightarrow \quad c_1 + c_2 + c_3 = 0$

(t=1) $\quad (P^1)_{13} = 0.2 \quad \Rightarrow \quad c_1 + 0.4c_2 - 0.6c_3 = 0.2$

6c cont.) For $t=2$, we need to find $(P^2)_{13}$:

$$P^2 = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & 0.2 \\ \cdot & \cdot & 0.6 \\ \cdot & \cdot & 0.2 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & 0.28 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

So $(t=2)$ $\quad (P^2)_{13} = 0.28 = c_1 + (0.4)^2 c_2 + (-0.6)^2 c_3$

The initial equations are therefore:

$$c_1 + c_2 + c_3 = 0 \quad ①$$

$$c_1 + 0.4 c_2 - 0.6 c_3 = 0.2 \quad ②$$

$$c_1 + 0.16 c_2 + 0.36 c_3 = 0.28 \quad ③$$

Using $c_1 = -(c_2 + c_3)$ from ① in ②, ③ $\Rightarrow$

② $\quad -c_2 - c_3 + 0.4 c_2 - 0.6 c_3 = 0.2$

$$\Rightarrow \quad -0.6 c_2 - 1.6 c_3 = 0.2 \quad (2*)$$

③ $\quad -c_2 - c_3 + 0.16 c_2 + 0.36 c_3 = 0.28$

$$\Rightarrow \quad -0.84 c_2 - 0.64 c_3 = 0.28 \quad (3*)$$

$(2*) - 2.5 * (3*) \Rightarrow \quad 1.5 c_2 = -0.5 \Rightarrow \underline{\underline{c_2 = -\frac{1}{3}}}$

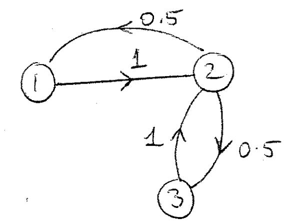
In $(2*) \Rightarrow \underline{\underline{c_3 = 0.}}$

Thus $\quad \underline{\underline{c_1 = \frac{1}{3}, \quad c_2 = -\frac{1}{3}, \quad c_3 = 0.}}$

The general solution is therefore:

$$\underline{\underline{(P^t)_{13} = \frac{1}{3} - \frac{1}{3}(0.4)^t}} \quad \text{for } t = 0, 1, 2, \ldots$$

7) a)



b) Equilibrium distribution $\underset{\sim}{\pi}$ satisfies $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$

and $\sum_{i=1}^{3} \pi_i = 1$.

Now $\quad (\pi_1 \ \ \pi_2 \ \ \pi_3) \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix} = (\pi_1 \ \ \pi_2 \ \ \pi_3)$

$$\Rightarrow \quad \begin{aligned} 0.5\,\pi_2 &= \pi_1 && ① \\ \pi_1 + \pi_3 &= \pi_2 && ② \\ 0.5\,\pi_2 &= \pi_3 && ③ \end{aligned}$$

①, ③ $\Rightarrow \pi_1 = \pi_3$, so ② $\Rightarrow \pi_2 = 2\pi_1$.

Then $\quad \pi_1 + \pi_2 + \pi_3 = 1 \quad \Rightarrow \quad \pi_1 + 2\pi_1 + \pi_1 = 1$

$$\Rightarrow \quad \underline{\pi_1 = \frac{1}{4}.}$$

So $\quad \underline{\underline{\underset{\sim}{\pi}^T = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)}}$ is an equilibrium distribution for P.

c) The matrix P is irreducible, but it is <u>not</u> aperiodic (period of all states is 2).

Thus the chain does <u>not</u> converge to the equilibrium dist.^{n} $\underset{\sim}{\pi}$.

8) a) Consider $\underset{\sim}{h} = (h_0, \dots, h_{10})$, where h_k is the probability of hitting state 10, given that gambler starts at state k.

The hitting probabilities are the minimal non-negative solutions to the equations:

$$h_{10} = 1$$
$$h_k = \sum_{j=0}^{10} p_{kj} h_j \quad \text{for } k \neq 10.$$

From the diagram, state 0 is an absorbing state, so $h_0 = 0$.

Otherwise, $h_k = \frac{1}{2} h_{k-1} + \frac{1}{2} h_{k+1}$. $(k = 1, 2, \dots, 9)$

So $h_1 = \frac{1}{2} h_0 + \frac{1}{2} h_2 = \frac{1}{2} * 0 + \frac{1}{2} h_2$

$\Rightarrow \underline{h_1 = \frac{1}{2} h_2}$

Also, $h_2 = \frac{1}{2} h_1 + \frac{1}{2} h_3 = \frac{1}{2} \cdot \frac{1}{2} h_2 + \frac{1}{2} h_3$

$\Rightarrow \frac{3}{4} h_2 = \frac{1}{2} h_3$

$\Rightarrow \underline{h_2 = \frac{2}{3} h_3}$

The hypothesis that $h_k = \frac{k}{k+1} h_{k+1}$ is true for $k = 1, 2$.

Prove by induction.
Suppose it is true up to $k-1$:

Then $h_k = \frac{1}{2} h_{k-1} + \frac{1}{2} h_{k+1}$

$= \frac{1}{2} \cdot \frac{k-1}{k} h_k + \frac{1}{2} h_{k+1}$

$\Rightarrow h_k \left(1 - \frac{k-1}{2k}\right) = \frac{1}{2} h_{k+1}$

$h_k \frac{(k+1)}{2k} = \frac{1}{2} h_{k+1}$

$\Rightarrow \underline{h_k = \left(\frac{k}{k+1}\right) h_{k+1}}$

So the hypothesis is true for k
$\Rightarrow$ proof by induction.

8 a cont) Thus $h_0 = 0$, $h_{10} = 1$,

and $h_k = \left(\frac{k}{k+1}\right) h_{k+1}$ for $k = 1, \dots, 9$.

b) We have
$$h_{10} = 1$$
$$h_9 = \frac{9}{10} h_{10} = \frac{9}{10}$$
$$h_8 = \frac{8}{9} \cdot \frac{9}{10} = \frac{8}{10}$$
$$h_7 = \frac{7}{8} \cdot \frac{8}{10} = \frac{7}{10}$$
$$\vdots$$

Clearly, $\underline{h_1 = \frac{1}{10}}$ i.e. $\underline{\mathbb{P}(\text{win } \$10, \text{ starting at } \$1) = \frac{1}{10}}$.

c) The gambler might never reach \$10, starting from \$1.

Thus, $\underline{\mathbb{E}(T_1) = \infty}$.

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9) X_t = maximum score after t tosses $\Rightarrow$ X_t can be $1, 2, \dots, 6$.

<u>Suppose $X_t = 1$</u> (i.e. maximum score is currently 1):
Then the next toss gives maximum $1, 2, \dots, 6$ with equal prob,

i.e. $\mathbb{P}(X_{t+1} = k \mid X_t = 1) = \frac{1}{6}$ for $k = 1, \dots, 6$.

<u>Suppose $X_t = 2$</u>:

the maximum on the next toss will remain 2 if the next toss is 1 or 2, (prob. $\frac{2}{6}$), otherwise it is 3, 4, 5, or 6 each with probability $\frac{1}{6}$.

<u>Suppose $X_t = k$</u>:

The maximum on the next toss is k with prob. $\frac{k}{6}$ (if $1, 2, \dots, k$ is thrown), otherwise it is $k+1, k+2, \dots, 6$ each with prob. $\frac{1}{6}$.

9 cont.) Thus the transition matrix is:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

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10) Possibilities:

Person on Floor 1	Person on Floor 2	X (first stop)	Probability
going → 2 (prob $\frac{1}{3}$)	→ 3 (prob $\frac{1}{8}$)	2	$\frac{1}{3} * \frac{1}{8} = \frac{1}{24}$
→ 2 (" $\frac{1}{3}$)	→ 1 (" $\frac{7}{8}$)	2	$\frac{1}{3} * \frac{7}{8} = \frac{7}{24}$
→ 3 (" $\frac{2}{3}$)	→ 3 (" $\frac{1}{8}$)	2	$\frac{2}{3} * \frac{1}{8} = \frac{2}{24}$
→ 3 (" $\frac{2}{3}$)	→ 1 (" $\frac{7}{8}$)	3	$\frac{2}{3} * \frac{7}{8} = \frac{14}{24}$

Thus X has the following distribution:

x	2	3
$\mathbb{P}(X = x)$	$\frac{10}{24}$	$\frac{14}{24}$