THE UNIVERSITY OF AUCKLAND

MOCK EXAMINATION
Campus: City

STATISTICS

Stochastic Processes

(Time allowed: THREE hours)

NOTE: Attempt **ALL** questions. Marks for each question are shown in brackets.
An Attachment containing useful information is found on page 4.

1. Let A and B be any events.

   (a) Show that $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(\overline{A} \cap \overline{B}) - 1$. (4 M)

   (b) Starting from the result in part (a), show that $\mathbb{P}(A \cup B) = 1 - \mathbb{P}(\overline{A} \cap \overline{B})$. (2 E)

2. (a) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common probability generating function $G_X(s)$. Let N be a random variable, independent of the X_i's, with PGF $G_N(s)$, and let $T = X_1 + \ldots + X_N$ (the sum of a random number of random variables). Show that the PGF of T is:

$$G_T(s) = G_N\Big(G_X(s)\Big).$$

   (5 M)

   (b) Using the result above, find an expression for $\mathbb{E}(T)$. (3 E)

   (c) For $i = 1, 2, \ldots$, let

$$X_i = \begin{cases} 1 & \text{with probability } \alpha, \\ 0 & \text{with probability } 1 - \alpha. \end{cases}$$

   Find $G_X(s)$, the common PGF of $X_1, X_2, \ldots$. (2 E)

   (d) Suppose that $N \sim \text{Binomial}(n, p)$. Using the list of probability generating functions in the Attachment, and the results from parts (a) and (c), find the PGF of $T = X_1 + \ldots + X_N$. Hence name the distribution of T. (5 E)

3. Dr Fraudster is trying to phone the Inland Revenue department about a tax statement. The call is connected immediately with probability 0.1; otherwise, she is put on hold for 30 seconds. Every 30 seconds, the call is connected with probability 0.1, independently of previous occasions; otherwise she has to wait another 30 seconds before the next opportunity. Let T be the total time in seconds that Dr Fraudster has to wait before the call is connected. Find $\mathbb{E}(T)$. (4 E)

4. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the population size at time n, and $Z_0 = 1$. Let Y be the family size distribution. Suppose that $Y \sim \text{Geometric}(p = 0.5)$, so that

$$\mathbb{P}(Y = y) = \left(\frac{1}{2}\right)^{y+1} \qquad \text{for } y = 0, 1, 2, \ldots$$

   (a) Let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. Show that

$$G(s) = \frac{1}{2 - s},$$

   and state the radius of convergence. (4 E)

   (b) Let $G_n(s) = \mathbb{E}\left(s^{Z_n}\right)$ be the PGF of the population size at time n. Prove by mathematical induction that, when $Y \sim \text{Geometric}(0.5)$,

$$G_n(s) = \frac{n - (n-1)s}{(n+1) - ns}.$$

   (You may assume without proof any general expressions for $G_n(s)$.) (6 M)

   (c) Find the probability that the population has become extinct *by* generation 10. (2 M)

   (d) Find the probability that the population becomes extinct *at* generation 6. (3 M)

   (e) Find the probability of eventual extinction, γ. (3 E)

5. Three political parties, A, B, and C, are getting ready for a general election. Every Monday, an opinion poll is taken to find out which party is in the lead. During the week, the leading party suffers a political crisis with probability α, independently of what happens in other weeks. If the party suffers a crisis, the next poll will rank it last, although the other two parties remain in the same order. If the leading party does not suffer a crisis, it remains in the lead the following week; in that case, the other two parties remain in the same order with probability 0.5, or swap order with probability 0.5.

   For example, if the ranking in week t is A, B, C, then the ranking in week $t + 1$ is:

   - B, C, A if there is a crisis;
   - A, B, C or A, C, B with equal probability if there is no crisis.

   (a) Let $X_0, X_1, \ldots$ be a Markov chain, where X_t represents the ranking of party A in week t. Thus X_t can take values 1 (if party A is in the lead), 2, and 3 (if party A is last). Find the transition matrix of the Markov chain. (6 M)

   (b) Using your answer to part (a), draw the transition diagram of the Markov chain. (2 E)

   (c) Find whether the Markov chain converges to an equilibrium distribution as $t \to \infty$, and if so, find the equilibrium distribution. (5 M)

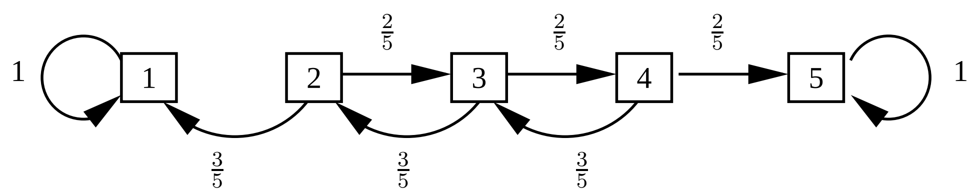
6. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3, 4, 5, 6\}$, with transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

   (a) Draw the transition diagram, and identify all communicating classes. For each class, state whether or not it is closed. (5 E)

   (b) Let the set $A = \{1, 2\}$. Find the vector of hitting probabilities, $\boldsymbol{h_A} = (h_{1A}, \ldots, h_{6A})^T$, where h_{iA} is the probability of eventually hitting the set A, starting from state i. (6 M)

7. Let $\{X_0, X_1, \ldots\}$ be a Markov chain with transition diagram below. Let the set $A = \{1, 5\}$.



   (a) Define the random variable T_k to be the number of steps taken to hit set A, starting from state k, for $k = 1, \ldots, 5$. Let $G_k(s) = \mathbb{E}(s^{T_k})$ be the probability generating function of T_k. By conditioning on the outcome of the first step, and using ideas of conditional expectation, show that

$$G_3(s) = \frac{13s^2}{25 - 12s^2}.$$

(8 H)

   (b) Show that T_3 is **not** a defective random variable. (1 M)

   (c) Using the expression for $G_3(s)$ above, find $\mathbb{E}(T_3)$. (4 M)

   (d) The PGF $G_3(s)$ could help us to find the whole distribution of T_3, but differentiating $G_3(s)$ repeatedly soon becomes difficult. Instead, use ideas of conditional probability, conditioning on the outcome of the first step, to show that:

$$\mathbb{P}(T_3 = 2) = \frac{13}{25} \qquad \mathbb{P}(T_3 = 4) = \frac{12 \times 13}{25^2} \qquad \mathbb{P}(T_3 = 6) = \frac{12^2 \times 13}{25^3}.$$

(10 H)

   (e) Based on the probabilities given above, form an inductive hypothesis for $\mathbb{P}(T_3 = 2n)$ ($n = 1, 2, \ldots$), and prove it. Hence calculate $\mathbb{E}(T_3)$. (Use part (c) to check your answer.)
   [Hint: you may use the formula for the expectation of a Geometric distribution given in the Attachment.] (10 H)

ATTACHMENT FOLLOWS

ATTACHMENT

1. **Discrete Probability Distributions**

Distribution	$\mathbb{P}(X = x)$	$\mathbb{E}(X)$	PGF, $\mathbb{E}(s^X)$
Geometric(p)	pq^x (where $q = 1 - p$), for $x = 0, 1, 2, \ldots$	$\frac{q}{p}$	$\frac{p}{1 - qs}$
	Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Binomial(n, p)	$\binom{n}{x} p^x q^{n-x}$ (where $q = 1 - p$) for $x = 0, 1, 2, \ldots, n$.	np	$(ps + q)^n$
	Number of successes in n independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Poisson(λ)	$\frac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \ldots$	λ	$e^{\lambda(s-1)}$

2. **Uniform Distribution:** $X \sim \text{Uniform}(a, b)$.
Probability density function, $f_X(x) = \frac{1}{b-a}$ for $a < x < b$. Mean, $\mathbb{E}(X) = \frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

Definition:	$G_X(s) = \mathbb{E}(s^X)$	
Moments:	$\mathbb{E}(X) = G'_X(1)$	$\mathbb{E}\left\{X(X-1)\ldots(X-k+1)\right\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \frac{1}{n!} G_X^{(n)}(0)$	

4. **Geometric Series:** $1 + r + r^2 + r^3 + \ldots \quad = \quad \sum_{x=0}^{\infty} r^x \quad = \quad \frac{1}{1 - r}$ for $|r| < 1$.

Finite sum: $\sum_{x=0}^{n} r^x = \frac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p, q \in \mathbb{R}$, and integer n, $(p + q)^n \quad = \quad \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$.