Stats 325 Mock 2 Solutions

1a) $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(\bar{A} \cap \bar{B}) - 1$ to prove.

$$\begin{aligned}
RHS &= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(\bar{A} \mid \bar{B})\mathbb{P}(\bar{B}) - 1 \\
&= \mathbb{P}(A) + \mathbb{P}(B) + (1 - \mathbb{P}(A \mid \bar{B}))\mathbb{P}(\bar{B}) - 1 \\
&= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(\bar{B}) - \mathbb{P}(A \cap \bar{B}) - 1 \\
&= \mathbb{P}(A) - \mathbb{P}(A \cap \bar{B}) \quad \text{because } \mathbb{P}(B) + \mathbb{P}(\bar{B}) = 1 \\
&= \mathbb{P}(A \cap B) \quad \text{by the Partition Rule, because } \mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \bar{B}).
\end{aligned}$$

$\underline{RHS = LHS.}$

b) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$= \underline{1 - \mathbb{P}(\bar{A} \cap \bar{B})}$ from part (a) as required.

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2) a)

$$\begin{aligned}
G_T(s) &= \mathbb{E}(s^T) \\
&= \mathbb{E}(s^{X_1 + \dots + X_N}) \\
&= \mathbb{E}_N\{\mathbb{E}(s^{X_1 + \dots + X_N} \mid N)\} \quad \text{(conditional expectation)} \\
&= \mathbb{E}_N\{\mathbb{E}(s^{X_1})\mathbb{E}(s^{X_2})\dots\mathbb{E}(s^{X_N})\}
\end{aligned}$$

because the X_i's are independent of each other and of N

$$\begin{aligned}
&= \mathbb{E}\{\mathbb{E}(s^{X_1})^N\} \quad \text{because the } X_i\text{'s are identically distributed} \\
&= \mathbb{E}\{G_X(s)^N\}
\end{aligned}$$

$\underline{G_T(s) = G_N(G_X(s))}$ as required.

2. b). $\mathbb{E}(T) = G_T'(1)$

Now $G_T'(s) = G_N'(G_X(s))\,G_X'(s)$

But $G_X(1) = 1$ for any non-defective r.v. X

and $G_N'(1) = \mathbb{E}N, \quad G_X'(1) = \mathbb{E}X.$

So

$$\begin{aligned}
\mathbb{E}(T) &= G_T'(1) \\
&= G_N'(G_X(1))\,G_X'(1) \\
&= G_N'(1)\,\mathbb{E}X
\end{aligned}$$

$$\underline{\mathbb{E}T = (\mathbb{E}N)(\mathbb{E}X)}$$

c) $G_X(s) = \mathbb{E}(s^{X_i}) = s^0\,\mathbb{P}(X_i = 0) + s^1\,\mathbb{P}(X_i = 1)$

$= (1-\alpha) + s\alpha$

So $\underline{G_X(s) = \alpha s + (1-\alpha)}$, for $s \in \mathbb{R}$.

d) $N \sim Bin(n, p)$, so $G_N(s) = (ps + q)^n$ where $q = 1-p$.

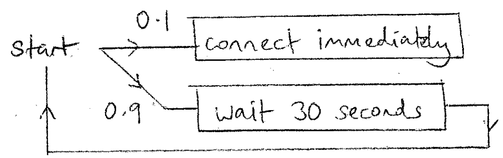
Thus

$$\begin{aligned}
G_T(s) &= G_N(G_X(s)) \quad \text{from (a)} \\
&= (p\,G_X(s) + q)^n \\
&= (p(\alpha s + (1-\alpha)) + q)^n \\
&= (p\alpha s + p - \alpha p + 1 - p)^n
\end{aligned}$$

$$\underline{G_T(s) = (\alpha p s + (1-\alpha p))^n}$$

Thus $\underline{T \sim Binomial(n, \alpha p)}$.

3)



$$\mathbb{E}(T) = 0.1 * 0 + 0.9\,(30 + \mathbb{E}(T))$$

$$\mathbb{E}T\,(1-0.9) = 0.9 * 30$$

$$\underline{\underline{\mathbb{E}T = 270 \text{ seconds}}} \quad (4.5 \text{ minutes}).$$

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4) a)
$$G(s) = \mathbb{E}(s^Y) = \sum_{y=0}^{\infty} s^y\, \mathbb{P}(Y=y)$$
$$= \sum_{y=0}^{\infty} s^y \left(\tfrac{1}{2}\right)^{y+1}$$
$$= \tfrac{1}{2} \sum_{y=0}^{\infty} \left(\tfrac{1}{2}s\right)^y$$
$$= \tfrac{1}{2} \cdot \frac{1}{1-\tfrac{1}{2}s}$$ for $\left|\tfrac{1}{2}s\right| < 1$: see sum of Geometric series in Attachment.

$$\underline{\underline{G(s) = \frac{1}{2-s}}}$$ as stated.

The radius of convergence is $\underline{\underline{2}}$, because we require $\left|\tfrac{1}{2}s\right| < 1$ $\Rightarrow -2 < s < 2$.

b) $G_n(s) = \mathbb{E}(s^{Z_n}) = G(G_{n-1}(s))$ (general result).

Now $G_1(s) = \mathbb{E}(s^{Z_1}) = \dfrac{1}{2-s}$.

$$G_2(s) = G(G_1(s)) = \frac{1}{2-G_1(s)}$$
$$= \frac{1}{2-\frac{1}{2-s}} = \frac{2-s}{3-2s}$$

4b cont.) Thus the hypothesis is true for $n=1$ and $n=2$.

Suppose that $$G_n(s) = \frac{n-(n-1)s}{(n+1)-ns}.$$

Then $$G_{n+1}(s) = G(G_n(s))$$
$$= \frac{1}{2-G_n(s)}$$
$$= \frac{1}{2-\frac{n-(n-1)s}{n+1-ns}}$$
$$= \frac{n+1-ns}{2(n+1)-2ns-n+(n-1)s}$$
$$= \frac{(n+1)-ns}{(n+2)-(2n-n+1)s}$$

$$\underline{\underline{G_{n+1}(s) = \frac{(n+1)-ns}{(n+2)-(n+1)s}}}$$

Thus the inductive hypothesis holds for $n+1$ too, so it is proved.

c) $$\mathbb{P}(\text{extinct by generation } 10) = \mathbb{P}(Z_{10}=0)$$
$$= G_{10}(0)$$
$$= \frac{10-9*0}{11-10*0}$$
$$= \underline{\underline{\frac{10}{11}}} \quad (0.909)$$

d) $$\mathbb{P}(\text{extinct } \underline{\text{at}} \text{ generation } 6) = \mathbb{P}(Z_6=0) - \mathbb{P}(Z_5=0)$$
$$= G_6(0) - G_5(0)$$

4d cont.) So $\mathbb{P}(\text{extinct at generation } 6) = \frac{6}{7} - \frac{5}{6}$

$= \frac{1}{42} \quad (0.0238).$

e) γ is the smallest solution ≥ 0 to $G(s) = s$.

Now $G(s) = \frac{1}{2-s} = s \implies s(2-s) = 1$

$\implies s^2 - 2s + 1 = 0$

$\implies (s-1)^2 = 0$

$\implies s = 1, \ s = 1.$

So $\gamma = \mathbb{P}(\text{extinction}) = 1.$

[Alternative : note that $\mathbb{E}(Y) = 1$ for $Y \sim \text{Geometric}(1/2)$, so $\gamma = 1.$]

5) a) Let X_t be the ranking of party A in week t.

If $X_t = 1$, then $X_{t+1} = \begin{cases} 3 & \text{if crisis (prob. } \alpha) \\ 1 & \text{if no crisis (prob. } 1-\alpha). \end{cases}$

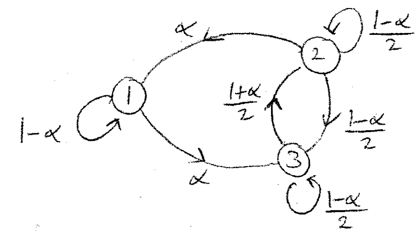
If $X_t = 2$, then $X_{t+1} = \begin{cases} 1 & \text{if leading party has crisis } (\alpha) \\ 2 \text{ or } 3 & \text{with prob. } \tfrac{1}{2} \text{ each if no crisis } (1-\alpha). \end{cases}$

If $X_t = 3$, then $X_{t+1} = \begin{cases} 2 & \text{if leading party has crisis } (\alpha) \\ 2 \text{ or } 3 & \text{with prob. } \tfrac{1}{2} \text{ each if no crisis } (1-\alpha). \end{cases}$

Thus the transition matrix is:

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1-\alpha & 0 & \alpha \\ \alpha & \frac{1}{2}(1-\alpha) & \frac{1}{2}(1-\alpha) \\ 0 & \alpha + \frac{1}{2}(1-\alpha) & \frac{1}{2}(1-\alpha) \end{pmatrix} \end{array} = \begin{pmatrix} 1-\alpha & 0 & \alpha \\ \alpha & \frac{1-\alpha}{2} & \frac{1-\alpha}{2} \\ 0 & \frac{1+\alpha}{2} & \frac{1-\alpha}{2} \end{pmatrix}.$$

b)



c) The chain is irreducible and aperiodic, so if an equilibrium distribution exists (it will for a finite state space), the chain will converge to the distribution.

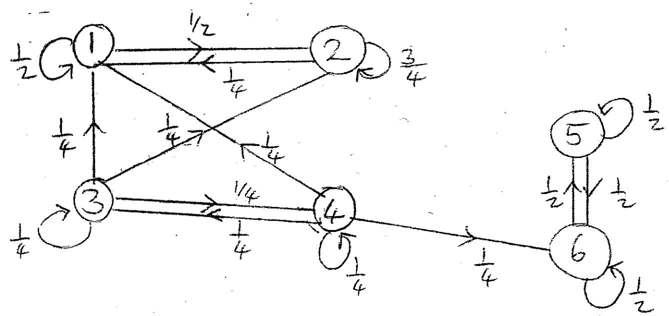
If $\underset{\sim}{\pi}^T = (\pi_1, \pi_2, \pi_3)$ is equilibrium for P, then $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$, $\pi_1 + \pi_2 + \pi_3 = 1$.

Thus $(1-\alpha)\pi_1 + \alpha\pi_2 = \pi_1 \implies \alpha(\pi_1 - \pi_2) = 0 \implies \pi_1 = \pi_2$

also $\left(\frac{1-\alpha}{2}\right)\pi_2 + \left(\frac{1+\alpha}{2}\right)\pi_3 = \pi_2 \implies (1+\alpha)(\pi_2 - \pi_3) = 0 \implies \pi_2 = \pi_3$

Thus $\pi_1 = \pi_2 = \pi_3$, and therefore the Markov chain converges to the equilibrium distribution $\underset{\sim}{\pi} = \left(\tfrac{1}{3} \ \tfrac{1}{3} \ \tfrac{1}{3}\right).$

6) a)



Communicating classes:

$\{1, 2\}$ (closed); $\{3, 4\}$ (not closed);

$\{5, 6\}$ (closed).

b) $\underset{\sim}{h}_A$ is the minimum non-negative solution to:

$$h_{1A} = 1$$
$$h_{2A} = 1$$

$$h_{iA} = \sum_{j=1}^{6} p_{ij} h_{jA} \Rightarrow h_{3A} = \tfrac{1}{4}(h_{1A} + h_{2A} + h_{3A} + h_{4A})$$
$$= \tfrac{1}{4}(1 + 1 + h_{3A} + h_{4A})$$
$$\Rightarrow \tfrac{3}{4} h_{3A} = \tfrac{1}{2} + \tfrac{1}{4} h_{4A} \quad (*)$$

and $h_{4A} = \tfrac{1}{4}(h_{1A} + h_{3A} + h_{4A} + h_{6A})$

Now by inspection, $h_{5A} = h_{6A} = 0$; so

$$h_{4A} = \tfrac{1}{4}(1 + h_{3A} + h_{4A} + 0)$$
$$\Rightarrow \tfrac{3}{4} h_{4A} = \tfrac{1}{4} + \tfrac{1}{4} h_{3A}$$
$$\Rightarrow 3 h_{4A} = 1 + \tfrac{1}{3} * \tfrac{4}{2} + \tfrac{1}{3} * \tfrac{4}{4} h_{4A} \quad \text{from } (*)$$
$$\tfrac{8}{3} h_{4A} = \tfrac{5}{3} \Rightarrow \underline{\underline{h_{4A} = \tfrac{5}{8}}}$$

Thus $\underline{\underset{\sim}{h}_A = \left(1, 1, \tfrac{7}{8}, \tfrac{5}{8}, 0, 0\right)}$ (using h_{3A} from $(*)$).

7) a) $G_k(s) = \mathbb{E}(s^{T_k})$

Now $T_1 \equiv 0$ (constant) $\Rightarrow G_1(s) = \mathbb{E}(s^{T_1}) = 1$ (1)
and $T_5 \equiv 0$ (constant) $\Rightarrow G_5(s) = \mathbb{E}(s^{T_5}) = 1$. (2)

Consider $G_3(s) = \mathbb{E}(s^{T_3})$

But
$$T_3 = \begin{cases} 1 + T_4 & \text{with probability } 2/5 \quad (\text{first step is up}) \\ 1 + T_2 & \text{" " } 3/5 \quad (\text{" " down}) \end{cases}$$

So $G_3(s) = \mathbb{E}(s^{T_3})$
$$= \tfrac{2}{5}\mathbb{E}(s^{1+T_4}) + \tfrac{3}{5}\mathbb{E}(s^{1+T_2})$$
$$= \tfrac{s}{5}\left\{2\mathbb{E}(s^{T_4}) + 3\mathbb{E}(s^{T_2})\right\}$$
$$\underline{G_3(s) = \tfrac{s}{5}\left\{2G_4(s) + 3G_2(s)\right\}} \quad \text{by definitions.} \quad (3)$$

Similarly, $G_4(s) = \mathbb{E}(s^{T_4})$
$$= \tfrac{2}{5}\mathbb{E}(s^{1+T_5}) + \tfrac{3}{5}\mathbb{E}(s^{1+T_3})$$
$$\Rightarrow G_4(s) = \tfrac{s}{5}\left\{2G_5(s) + 3G_3(s)\right\}$$
$$\underline{G_4(s) = \tfrac{s}{5}\left\{2 + 3G_3(s)\right\}} \quad (\text{from } (2)). \quad (4)$$

Also, $G_2(s) = \mathbb{E}(s^{T_2})$
$$= \tfrac{2}{5}\mathbb{E}(s^{1+T_3}) + \tfrac{3}{5}\mathbb{E}(s^{1+T_1})$$
$$= \tfrac{s}{5}\left\{2G_3(s) + 3G_1(s)\right\}$$
$$\underline{G_2(s) = \tfrac{s}{5}\left\{2G_3(s) + 3\right\}} \quad (\text{from } (1)) \quad (5)$$

7a cont.) Substituting (4) and (5) in (3):

$$G_3(s) = \frac{2s}{5} G_4(s) + \frac{3s}{5} G_2(s)$$

$$= \frac{2s}{5} \cdot \frac{s}{5} \{2 + 3G_3(s)\} + \frac{3s}{5} \cdot \frac{s}{5} \{2G_3(s) + 3\}$$

$$= \frac{4s^2}{25} + \frac{9s^2}{25} + G_3(s) \left\{ \frac{6s^2}{25} + \frac{6s^2}{25} \right\}$$

$$G_3(s) = \frac{13 s^2}{25} + \frac{12 s^2}{25} G_3(s)$$

Thus $\left(1 - \frac{12 s^2}{25}\right) G_3(s) = \frac{13 s^2}{25}$

$$\Rightarrow \quad G_3(s) = \frac{13 s^2}{25\left(1 - \frac{12}{25} s^2\right)}$$

$$G_3(s) = \frac{13 s^2}{25 - 12 s^2} \quad \text{as required.}$$

b) T_3 is defective if and only if $G_3(1) < 1$.

Now $G_3(1) = \frac{13}{25-12} = 1$.

So T_3 is not defective.

c) $\mathbb{E}(T_3) = G_3'(1)$.

Now $G_3(s) = 13 s^2 (25 - 12 s^2)^{-1}$

$$\Rightarrow \quad G_3'(s) = 13 \cdot 2s\, (25 - 12s^2)^{-1} - 13 s^2 (25 - 12 s^2)^{-2} (-12)(2s)$$

$$= 26 s\, (25 - 12 s^2)^{-1} + 312\, s^2 (25 - 12 s^2)^{-2}$$

So

$$\mathbb{E}(T_3) = G_3'(1) = \frac{26}{13} + \frac{312}{13^2}$$

$$\mathbb{E}(T_3) = \frac{50}{13} \quad (3.85).$$

7d) $\mathbb{P}(T_3 = 2) = \frac{2}{5} \mathbb{P}(T_3 = 2 \mid \text{first step} \to 4)$

$$+ \frac{3}{5} \mathbb{P}(T_3 = 2 \mid \text{first step} \to 2)$$

$$= \frac{2}{5} \mathbb{P}(T_4 = 1) + \frac{3}{5} \mathbb{P}(T_2 = 1)$$

because one step used to get to state 4, $\Rightarrow$ 1 step left to hit A, if $T_3 = 2$.

(similar)

But $\mathbb{P}(T_4 = 1) = \frac{2}{5}$ (the only way of getting from 4 to A in one step.)

Similarly, $\mathbb{P}(T_2 = 1) = \frac{3}{5}$.

Thus $\mathbb{P}(T_3 = 2) = \frac{2}{5} * \frac{2}{5} + \frac{3}{5} * \frac{3}{5}$

$\therefore$ $\mathbb{P}(T_3 = 2) = \frac{13}{25}$ as stated. (*)

Now $\mathbb{P}(T_3 = 4) = \frac{2}{5} \mathbb{P}(T_4 = 3) + \frac{3}{5} \mathbb{P}(T_2 = 3)$ (**) by the same arguments.

But $\mathbb{P}(T_4 = 3) = \frac{2}{5} * 0 + \frac{3}{5} * \mathbb{P}(T_3 = 2)$

hit A immediately so cannot take 2 more steps

Similarly, $\mathbb{P}(T_2 = 3) = \frac{3}{5} * 0 + \frac{2}{5} * \mathbb{P}(T_3 = 2)$

Also, $\mathbb{P}(T_3 = 2) = \frac{13}{25}$ from (*).

So $\mathbb{P}(T_3 = 4) = \frac{2}{5}\left\{\frac{3}{5} * \frac{13}{25}\right\} + \frac{3}{5}\left\{\frac{2}{5} * \frac{13}{25}\right\}$ in (**)

$$\mathbb{P}(T_3 = 4) = \frac{12 * 13}{(25)^2} \quad \text{as stated. (***)}$$

7d cont.) Finally, $\mathbb{P}(T_3=6) = \frac{2}{5}\mathbb{P}(T_4=5) + \frac{3}{5}\mathbb{P}(T_2=5)$

so $\mathbb{P}(T_3=6) = \frac{2}{5}\left\{\frac{3}{5}\mathbb{P}(T_3=4)\right\} + \frac{3}{5}\left\{\frac{2}{5}\mathbb{P}(T_3=4)\right\}$

$= \frac{12}{25}\mathbb{P}(T_3=4)$

$$\underline{\underline{\mathbb{P}(T_3=6) = \frac{12^2 * 13}{25^3}}}$$ as stated (from $***$).

e) Inductive hypothesis: $\underline{\underline{\mathbb{P}(T_3=2n) = \frac{12^{n-1} * 13}{25^n}}}$

True for $n=1,2,3$ as in part (d).

Suppose it is true for n; i.e. $\mathbb{P}(T_3=2n) = \frac{12^{n-1} * 13}{25^n}$.

Then
$$\begin{aligned}
\mathbb{P}(T_3=2(n+1)) &= \tfrac{2}{5}\mathbb{P}(T_4=2n+1) + \tfrac{3}{5}\mathbb{P}(T_2=2n+1) \\
&= \tfrac{2}{5}\left\{\tfrac{3}{5}\mathbb{P}(T_3=2n)\right\} + \tfrac{3}{5}\left\{\tfrac{2}{5}\mathbb{P}(T_3=2n)\right\} \\
&= \tfrac{12}{25}\mathbb{P}(T_3=2n) \\
&= \tfrac{12}{25} * \frac{12^{n-1} * 13}{25^n} \quad \text{by inductive hypothesis} \\
&= \underline{\underline{\frac{12^n * 13}{25^{n+1}}}}
\end{aligned}$$

Thus the inductive hypothesis holds for $n+1$, so it is proved.

Now $\mathbb{E}(T_3) = \sum_{n=0}^{\infty} n\,\mathbb{P}(T_3=n)$.

But T_3 can take only even values (by inspection of the transition diagram). So

$$\begin{aligned}
\mathbb{E}(T_3) &= \sum_{n=1}^{\infty} 2n\,\mathbb{P}(T_3=2n) \\
&= 2\sum_{n=1}^{\infty} n * \frac{12^{n-1} * 13}{25^n}
\end{aligned}$$

7e cont) $\mathbb{E}(T_3) = \frac{2}{12} * 13 \sum_{n=1}^{\infty} n * \left(\frac{12}{25}\right)^n$ $\quad (+)$

But if a r.v. $X \sim \text{Geometric}\left(p = 1-\frac{12}{25}\right)$, then

$$\mathbb{E}X = \sum_{n=1}^{\infty} np(1-p)^n = \frac{1-p}{p}$$ from Attachment.

So $\left(1-\frac{12}{25}\right)\sum_{n=1}^{\infty} n\left(\frac{12}{25}\right)^n = \frac{12/25}{1-12/25}$

$$\Rightarrow \sum_{n=1}^{\infty} n\left(\frac{12}{25}\right)^n = \frac{12/25}{(1-12/25)^2} = \frac{12 * 25}{13^2}$$

Thus from $(+)$,
$$\begin{aligned}
\mathbb{E}(T_3) &= \frac{2}{12} * 13 * \frac{12 * 25}{13^2} \\
&= \frac{2 * 25}{13}
\end{aligned}$$

$$\underline{\underline{\mathbb{E}(T_3) = \frac{50}{13}}}$$ as in part (c).