THE UNIVERSITY OF AUCKLAND

MOCK EXAMINATION
Campus: City

STATISTICS

Stochastic Processes

(Time allowed: THREE hours)

NOTE: Attempt **ALL** questions. Marks for each question are shown in brackets.
An Attachment containing useful information is found on page 4.

1. (a) Let A, B, and C be any events, with $\mathbb{P}(B \cap C) \neq 0$. Show that

$$\mathbb{P}(A \mid B \cap C)\, \mathbb{P}(B \mid C) = \mathbb{P}(A \cap B \mid C).$$

(2 E)

(b) Let A and B be events, with $\mathbb{P}(A) = \frac{3}{4}$ and $\mathbb{P}(B) = \frac{1}{3}$. Show that

$$\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}.$$

[Hint: consider $\mathbb{P}(A \cup B)$.] (4 M)

2. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common probability generating function

$$G_X(s) = \left(\frac{s+1}{2}\right)^2.$$

Let N be a random variable, independent of the X_i's, with PGF

$$G_N(s) = \frac{1}{2-s}.$$

Let $T = X_1 + \ldots + X_N$ (the sum of a random number of random variables).

(a) Using $G_X(s)$, find the distribution of X: that is, find $\mathbb{P}(X = r)$ for $r = 0, 1, 2, \ldots$. (5 E)

(b) Find the PGF of T, $G_T(s)$. (5 M)

3. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the population size at time n, and $Z_0 = 1$. Let Y be the family size distribution, and suppose that Y has the following probability function:

y	0	1	2	3
$\mathbb{P}(Y = y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$

(a) Find the probability generating function of Y, $G(s)$. (2 E)

(b) Find $\mathbb{P}(Z_2 = 0)$. (2 E)

(c) Find the probability of eventual extinction, γ. (4 E)

4. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, with probability generating function $G(s)$. Let $M^{(n)}$ be the **total progeny up to generation n**: that is,

$$M^{(n)} = Z_0 + Z_1 + Z_2 + \ldots + Z_n.$$

Let the probability generating function of $M^{(n)}$ be $J_n(s) = \mathbb{E}\left(s^{M^{(n)}}\right)$.

(a) Show that $J_n(s) = sG(J_{n-1}(s))$ for $n = 1, 2, \ldots$. (8 H)

(b) Find $\mathbb{P}(M^{(n)} = 0)$ for any n. (1 E)

(c) Now suppose that $Y \sim \text{Geometric}(p)$, so that $G(s) = \dfrac{p}{1 - qs}$, where $q = 1 - p$. Using the result from part (a), give a general recurrence relationship for $J_n(s)$ in terms of $J_{n-1}(s)$. (1 E)

(d) Using the expression for $J_n(s)$ from part (c), find a recurrence relationship for $\mathbb{E}\left(M^{(n)}\right)$ in terms of $\mathbb{E}\left(M^{(n-1)}\right)$. Solve the recurrence relationship to find a general expression for $\mathbb{E}\left(M^{(n)}\right)$ in terms of $\mu = \frac{q}{p}$. (8 H)

5. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3, 4\}$, with transition matrix

$$P = \begin{pmatrix} 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{pmatrix}.$$

(a) Draw the transition diagram. (2 E)

(b) Find an equilibrium distribution for P. (4 E)

(c) Does X_t converge to the distribution in (b) as $t \to \infty$? Explain why or why not. (3 M)

6. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}, \quad \text{where } 0 < \alpha < 1, \;\; 0 < \beta < 1.$$

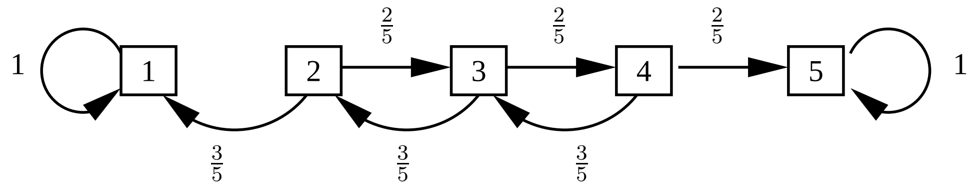
(a) Suppose that $X_0 \sim (0.2, 0.8)^T$. Find a vector describing the distribution of X_1. (2 E)

(b) Find $\mathbb{P}(X_1 = 1, X_2 = 1, X_3 = 2 \,|\, X_0 = 2)$. (2 E)

(c) **NOT EXAMINABLE FROM 2006 ONWARDS.**
(Find a general formula for the matrix P^t, for any t.) (12 M)

(d) Tom likes watching TV. There are 6 TV channels he can watch. Every time he switches on the TV, he leaves it on the same channel as last time with probability 0.4, or changes channel with probability 0.6. If he changes channel, he is equally likely to choose any of the alternative channels. Suppose Tom starts watching Channel 1 at time 0. What is the probability that he is watching Channel 1 at time 10? (6 M)

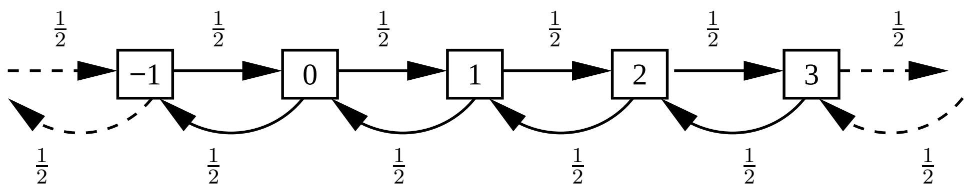
7. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with transition diagram below.



(a) What is the period of state 2? (2 E)

(b) Let set $A = \{1, 5\}$. Find the vector of expected hitting times, $\boldsymbol{m_A} = (m_{1A}, \ldots, m_{5A})^T$, where m_{iA} is the expected number of steps taken to hit the set A, starting from state i. (7 M)

8. Let $\{X_0, X_1, X_2, \ldots\}$ be a simple symmetric random walk on the integers, with transition diagram below.



Define T_{ij} to be the number of steps taken to reach state j, starting at state i, for any integers i and j.

Let $H(s) = \mathbb{E}(s^{T_{01}})$ be the probability generating function of T_{01}, the number of steps taken to reach state 1, starting from state 0.

(a) Let $G_{ij}(s) = \mathbb{E}(s^{T_{ij}})$, the PGF of T_{ij}. Express $G_{ij}(s)$ in terms of $H(s)$ for all integers i and j. Treat the cases $j > i$, $j < i$, and $j = i$ separately. (8 H)

(b) Show that

$$H(s) = \frac{1 - \sqrt{1 - s^2}}{s},$$

and state the radius of convergence. (10 H)

ATTACHMENT

1. **Discrete Probability Distributions**

Distribution	$\mathbb{P}(X = x)$	$\mathbb{E}(X)$	PGF, $\mathbb{E}(s^X)$
Geometric(p)	pq^x (where $q = 1 - p$), for $x = 0, 1, 2, \dots$	$\frac{q}{p}$	$\frac{p}{1 - qs}$
	Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Binomial(n, p)	$\binom{n}{x} p^x q^{n-x}$ (where $q = 1 - p$) for $x = 0, 1, 2, \dots, n$.	np	$(ps + q)^n$
	Number of successes in n independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Poisson(λ)	$\frac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \dots$	λ	$e^{\lambda(s-1)}$

2. **Uniform Distribution:** $X \sim \text{Uniform}(a, b)$.
Probability density function, $f_X(x) = \frac{1}{b-a}$ for $a < x < b$. Mean, $\mathbb{E}(X) = \frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

Definition:	$G_X(s) = \mathbb{E}(s^X)$	
Moments:	$\mathbb{E}(X) = G_X'(1)$	$\mathbb{E}\left\{X(X-1)\dots(X-k+1)\right\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \frac{1}{n!} G_X^{(n)}(0)$	

4. **Geometric Series:** $1 + r + r^2 + r^3 + \dots = \sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$ for $|r| < 1$.

Finite sum: $\sum_{x=0}^{n} r^x = \frac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p, q \in \mathbb{R}$, and integer n, $(p+q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$.