Stats 325 Mock 3 Solutions

1a) $$\mathbb{P}(A \mid B\cap C)\,\mathbb{P}(B \mid C) = \frac{\mathbb{P}(A\cap B\cap C)}{\mathbb{P}(B\cap C)} \cdot \frac{\mathbb{P}(B\cap C)}{\mathbb{P}(C)}$$

(by definitions)

$$= \frac{\mathbb{P}(A\cap B\cap C)}{\mathbb{P}(C)}$$

$$= \underline{\underline{\mathbb{P}(A\cap B \mid C)}} \quad \text{as required.}$$

b) Clearly, $(A\cap B)$ is a subset of both A and B, so

$\mathbb{P}(A\cap B) \leq \mathbb{P}(A) = 3/4$

and $\mathbb{P}(A\cap B) \leq \mathbb{P}(B) = 1/3$

Thus $\underline{\underline{\mathbb{P}(A\cap B) \leq \frac{1}{3}}}$ as required. ⓐ

Now consider $$\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A\cap B)$$
$$= \frac{3}{4} + \frac{1}{3} - \mathbb{P}(A\cap B)$$
$$= \frac{13}{12} - \mathbb{P}(A\cap B).$$

So $$1 \geq \mathbb{P}(A\cup B) \Rightarrow 1 \geq \frac{13}{12} - \mathbb{P}(A\cap B)$$

$$\Rightarrow \underline{\underline{\mathbb{P}(A\cap B) \geq \frac{13}{12} - 1 = \frac{1}{12}}} \quad ⓑ$$

ⓐ and ⓑ together give $\underline{\underline{\frac{1}{12} \leq \mathbb{P}(A\cap B) \leq \frac{1}{3}}}$ as required.

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2) a) $$G_X(s) = \left(\frac{s+1}{2}\right)^2$$

Then $\mathbb{P}(X=r) = \frac{1}{r!}G_X^{(r)}(0)$ (see Attachment).

So $\mathbb{P}(X=0) = G_X(0) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

Next, $G_X'(s) = 2\left(\frac{s+1}{2}\right)\cdot\frac{1}{2} = \frac{s+1}{2}$

$G_X''(s) = \frac{1}{2}$

2a cont.) and $G_X^{(r)}(s) \equiv 0$ for $r > 2$.

So we have: $\mathbb{P}(X=0) = \frac{1}{4}$

$\mathbb{P}(X=1) = \frac{1}{1!}G_X'(0) = \frac{1}{2}$

$\mathbb{P}(X=2) = \frac{1}{2!}G_X''(0) = \frac{1}{2!}\cdot\frac{1}{2} = \frac{1}{4}$

$\mathbb{P}(X=r) = 0$ for $r > 2$.

x	0	1	2
$\mathbb{P}(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b) $T = X_1 + \dots + X_N$ where N is random.

$$G_T(s) = \mathbb{E}(s^T) = \mathbb{E}_N\{\mathbb{E}(s^T \mid N)\}$$
$$= \mathbb{E}_N\{\mathbb{E}(s^{X_1+\dots+X_N} \mid N)\}$$
$$= \mathbb{E}_N\{\mathbb{E}(s^{X_1})^N\}$$

(because the X_i's are independent, identically distributed (as X_1), and independent of N)

So $$G_T(s) = \mathbb{E}_N\{G_X(s)^N\}$$
$$= G_N\{G_X(s)\} \quad \text{by definitions}$$
$$= \frac{1}{2 - G_X(s)} \quad \text{by def.}^{\text{n}} \text{ of } G_N$$

$$\underline{\underline{G_T(s) = \frac{1}{2 - \left(\frac{s+1}{2}\right)^2}}} \quad \text{" " " } G_X$$

3) a) $G(s) = \mathbb{E}(s^Y)$

$$= \frac{1}{4}s^0 + \frac{1}{2}s + \frac{1}{8}s^2 + \frac{1}{8}s^3$$

$$\underline{\underline{G(s) = \frac{1}{4} + \frac{1}{2}s + \frac{1}{8}s^2 + \frac{1}{8}s^3}}$$

b) $\mathbb{P}(Z_2 = 0) = G_2(0) = G(G(0))$.

Now $G(0) = \frac{1}{4}$

so $G(G(0)) = G(\frac{1}{4}) = 0.385 \quad \left(\frac{197}{512}\right)$.

So $\underline{\underline{\mathbb{P}(Z_2 = 0) = 0.385}}$.

c) γ is the smallest non-negative solution to $G(s) = s$.
Thus

$$0 = G(s) - s$$
$$= \frac{1}{4} + \frac{1}{2}s + \frac{1}{8}s^2 + \frac{1}{8}s^3 - s$$

$\Rightarrow \quad 0 = 2 - 4s + s^2 + s^3$ (multiplying by 8) ⊛

Know $(s-1)$ must be a factor (checked $s=1$ satisfies ⊛).

So $(s-1)(s^2 + 2s - 2) = 0$

$\Rightarrow \quad s = 1, \quad s = \dfrac{-2 \pm \sqrt{4+4*2}}{2} = -1 \pm \sqrt{1+2} = -1 \pm \sqrt{3}$

The smallest solution ≥ 0 is $s = -1 + \sqrt{3} = \underline{0.732}$.

So $\underline{\underline{\gamma = \mathbb{P}(\text{extinction}) = \sqrt{3} - 1 = 0.732}}$.

4a) $J_n(s) = \mathbb{E}(s^{M^{(n)}})$

But $M^{(n)} = Z_0 + Z_1 + \dots + Z_n$

$= Z_0 + \left(\begin{array}{l}\text{descendants to generation } n-1 \text{ of the} \\ Z_1 \text{ individuals in generation } 1\end{array}\right)$

$= Z_0 + \{ M_1^{(n-1)} + \dots + M_{Z_1}^{(n-1)} \}$

where each $M_i^{(n-1)}$ has the same distribution as $M^{(n-1)}$.

So $M^{(n)} = 1 + M_1^{(n-1)} + \dots + M_{Z_1}^{(n-1)}$

Thus $J_n(s) = \mathbb{E}(s^{M^{(n)}})$

$$= \mathbb{E}\left(s^{1 + M_1^{(n-1)} + \dots + M_{Z_1}^{(n-1)}}\right)$$

$$= s\,\mathbb{E}\left(s^{M_1^{(n-1)} + \dots + M_{Z_1}^{(n-1)}}\right)$$

$$= s\,\mathbb{E}_{Z_1}\left\{ \mathbb{E}\left(s^{M_1^{(n-1)} + \dots + M_{Z_1}^{(n-1)}} \,\middle|\, Z_1\right)\right\} \quad \text{conditional expectation}$$

$$= s\,\mathbb{E}_{Z_1}\left\{ \mathbb{E}(s^{M_i^{(n-1)}})^{Z_1} \right\}$$

because each $M_i^{(n-1)}$ is independent of the others and of Z_1, and $\mathbb{E}(s^{M_i^{(n-1)}}) = \mathbb{E}(s^{M_1^{(n-1)}})$ for all i.

Thus $J_n(s) = s\,\mathbb{E}_{Z_1}\left(J_{n-1}(s)^{Z_1}\right)$ because $\mathbb{E}(s^{M_1^{(n-1)}}) = J_{n-1}(s)$ by definition

$\therefore \quad \underline{\underline{J_n(s) = s\,G(J_{n-1}(s))}}$ by definition of $G(r)$.

as stated.

b) $\mathbb{P}(M^{(n)} = 0) = J_n(0)$
$= 0 * G(J_{n-1}(0))$

$\Rightarrow \underline{\underline{\mathbb{P}(M^{(n)} = 0) = 0}}$ for any n (because there is always an individual at time 0, so $M^{(n)} > 0 \ \forall n$.)

4c) $G(s) = \frac{p}{1-qs}$

$$\Rightarrow \quad J_n(s) = s \cdot \frac{p}{1-qJ_{n-1}(s)} = \frac{ps}{1-qJ_{n-1}(s)} \qquad \text{for } n=1,2,\dots$$

d) $\mathbb{E}(M^{(n)}) = J_n'(1)$

Now $J_n(s) = ps\left(1-qJ_{n-1}(s)\right)^{-1}$

$$\Rightarrow \quad J_n'(s) = ps(-1)\left(1-qJ_{n-1}(s)\right)^{-2}\cdot\left(-qJ_{n-1}'(s)\right) + p\left(1-qJ_{n-1}(s)\right)^{-1}$$

$$\Rightarrow \quad J_n'(1) = pq\,J_{n-1}'(1)\left(1-qJ_{n-1}(1)\right)^{-2} + p\left(1-qJ_{n-1}(1)\right)^{-1} \quad (*)$$

But $J_n'(1) = \mathbb{E}(M^{(n)})$, and $J_{n-1}'(1) = \mathbb{E}(M^{(n-1)})$

Also, $J_{n-1}(1) = 1$ because this is true for any non-defective random variable.

So from $(*)$,
$$\mathbb{E}(M^{(n)}) = pq\,\mathbb{E}(M^{(n-1)})(1-q)^{-2} + p(1-q)^{-1}$$
$$= pq\,\mathbb{E}(M^{(n-1)})\,p^{-2} + p\cdot p^{-1}$$
$$= \frac{q}{p}\,\mathbb{E}(M^{(n-1)}) + 1$$

So the recurrence relationship is:

$$\mathbb{E}(M^{(n)}) = 1 + \frac{q}{p}\,\mathbb{E}(M^{(n-1)}).$$

Solving the recurrence: $M^{(0)} = 1$, so $\mathbb{E}(M^{(0)}) = 1$.

$$\mathbb{E}(M^{(1)}) = 1 + \frac{q}{p}\mathbb{E}(M^{(0)}) = 1 + \frac{q}{p}$$
$$\mathbb{E}(M^{(2)}) = 1 + \frac{q}{p}\mathbb{E}(M^{(1)}) = 1 + \frac{q}{p}\left(1+\frac{q}{p}\right)$$
$$= 1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2.$$

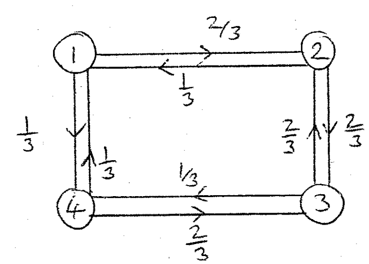
4d cont.) Continuing, claim that $\mathbb{E}(M^{(n)}) = 1 + \frac{q}{p} + \dots + \left(\frac{q}{p}\right)^n$.

Prove by induction: it is true for $n = 1, 2$; suppose it is true for n. Then
$$\mathbb{E}(M^{(n+1)}) = 1 + \frac{q}{p}\left\{1 + \frac{q}{p} + \dots + \left(\frac{q}{p}\right)^n\right\}$$
$$= 1 + \frac{q}{p} + \dots + \left(\frac{q}{p}\right)^{n+1} \Rightarrow \text{true for } n+1 \text{ too.}$$

So it follows that
$$\mathbb{E}(M^{(n)}) = 1 + \frac{q}{p} + \dots + \left(\frac{q}{p}\right)^n$$
$$\mathbb{E}(M^{(n)}) = 1 + \mu + \dots + \mu^n \qquad \text{in terms of } \mu = q/p.$$

Using the information in the Attachment, this gives:
$$\mathbb{E}(M^{(n)}) = \begin{cases} \dfrac{1-\mu^{n+1}}{1-\mu} & \text{if } \mu \neq 1 \\ n+1 & \text{if } \mu = 1. \end{cases}$$

5) a)



b) If $\underline{\pi}$ is an equilibrium distribution, then $\underline{\pi}$ satisfies
$$\underline{\pi}^T P = \underline{\pi}^T$$
$$\sum_{i=1}^{4} \pi_i = 1.$$

5b cont.) Solving:

$$\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T \Rightarrow \quad \tfrac{1}{3}(\pi_2 + \pi_4) = \pi_1 \quad (1)$$

$$\tfrac{2}{3}(\pi_1 + \pi_3) = \pi_2 \quad (2)$$

$$\tfrac{2}{3}(\pi_2 + \pi_4) = \pi_3 \quad (3)$$

$$\tfrac{1}{3}(\pi_1 + \pi_3) = \pi_4 \quad (4)$$

$$\sum_{i=1}^{4} \pi_i = 1 \Rightarrow \quad \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \quad (5)$$

Now (1), (3) $\Rightarrow \quad \pi_2 + \pi_4 = 3\pi_1 = \tfrac{3}{2}\pi_3$

$\Rightarrow \quad \underline{\pi_1 = \tfrac{1}{2}\pi_3}$

In eqn (2) $\Rightarrow \quad \tfrac{2}{3}\left(\tfrac{1}{2}\pi_3 + \pi_3\right) = \pi_2$

$\tfrac{2}{3} \cdot \tfrac{3}{2}\pi_3 = \pi_2$

$\Rightarrow \quad \underline{\pi_3 = \pi_2}$

Also, (2), (4) $\Rightarrow \quad \pi_1 + \pi_3 = \tfrac{3}{2}\pi_2 = 3\pi_4 \Rightarrow \underline{\pi_4 = \tfrac{1}{2}\pi_2}$

So $\underset{\sim}{\pi} = \left(\tfrac{1}{2}\pi_3,\ \pi_3,\ \pi_3,\ \tfrac{1}{2}\pi_3\right)$ (putting all together)

So $\sum_{i=1}^{4} \pi_i = 1 \Rightarrow \pi_3\left(\tfrac{1}{2} + 1 + 1 + \tfrac{1}{2}\right) = 1$

$$\Rightarrow \quad \underline{\underline{\pi_3 = \tfrac{1}{3}}}$$

The equilibrium distribution is $\underline{\underline{\underset{\sim}{\pi} = \left(\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{6}\right)}}$.

5c) The chain is irreducible, but every state has period 2, so it is not aperiodic.

Thus X_t does not converge to $\underset{\sim}{\pi}$ (or any distribution) as $t \to \infty$.

6) a) $X_0 \sim (0.2, 0.8)^T$

$$X_1 \sim (0.2,\ 0.8)\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

$$\Rightarrow \underline{\underline{X_1 \sim \left(0.2(1-\alpha) + 0.8\beta,\quad 0.2\alpha + 0.8(1-\beta)\right)}}$$

b) $\mathbb{P}(X_1 = 1, X_2 = 1, X_3 = 2 \mid X_0 = 2)$

$= \mathbb{P}(X_1 = 1 \mid X_0 = 2)\ \mathbb{P}(X_2 = 1 \mid X_1 = 1)\ \mathbb{P}(X_3 = 2 \mid X_2 = 1)$ by Markov Property

$= P_{21} * P_{11} * P_{12}$

$= \underline{\underline{\beta(1-\alpha)\alpha}}$

c) If P has distinct eigenvalues, $\lambda_1 = 1$ and λ_2, then a general formula for P^t is:

$$P^t = C_1 + C_2\lambda_2^t,$$

where C_1 and C_2 are 2x2 matrices to be determined.

Find eigenvalues: $\det(P - \lambda I) = 0$

$$\Rightarrow \det\begin{pmatrix} 1-\alpha-\lambda & \alpha \\ \beta & 1-\beta-\lambda \end{pmatrix} = 0$$

$$\Rightarrow 0 = (1-\alpha-\lambda)(1-\beta-\lambda) - \alpha\beta$$

$$= (1-\alpha)(1-\beta) - \lambda(1-\alpha+1-\beta) + \lambda^2 - \alpha\beta$$

$$= (1-\alpha-\beta) - \lambda(2-\alpha-\beta) + \lambda^2$$

$$= (\lambda-1)(\lambda-(1-\alpha-\beta))$$

6c) cont) Thus $\lambda = 1$ or $\lambda = 1-\alpha-\beta$.
The eigenvalues are distinct (for $\alpha > 0$ and $\beta > 0$),
so general solution is

$$\underline{P^t = C_1 + C_2 (1-\alpha-\beta)^t .}$$

Initial conditions :

$$P^0 = I \Rightarrow C_1 + C_2 = I \quad ①$$

$$P^1 = P \Rightarrow C_1 + C_2(1-\alpha-\beta) = P \quad ②$$

$$① - ② \Rightarrow C_2(\alpha+\beta) = I - P$$

$$\Rightarrow \underline{C_2 = \frac{1}{\alpha+\beta}(I-P)}$$

In ① :

$$C_1 = I - C_2 = \frac{1}{\alpha+\beta}\left((\alpha+\beta)I - I + P\right)$$

$$\underline{C_1 = \frac{1}{\alpha+\beta}\left((\alpha+\beta-1)I + P\right)}$$

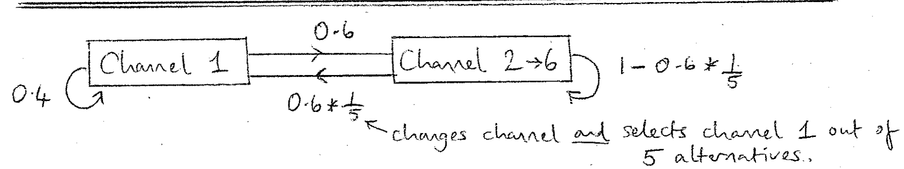
This gives finally :

$$P^t = \frac{1}{\alpha+\beta}\left\{(\alpha+\beta-1)I + P + (I-P)(1-\alpha-\beta)^t\right\}$$

$$= \frac{1}{\alpha+\beta}\left\{\begin{pmatrix} \alpha+\beta-1+1-\alpha & \alpha \\ \beta & \alpha+\beta-1+1-\beta \end{pmatrix} + \begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}(1-\alpha-\beta)^t\right\}$$

$$\underline{\underline{P^t = \frac{1}{\alpha+\beta}\left\{\begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} + \begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}(1-\alpha-\beta)^t\right\}}}$$

6d)



Put $\alpha = 0.6$, $\beta = 0.6 * \frac{1}{5}$, $t = 10$ in the formula from (c):
we want

$$\underline{(P^{10})_{11} = \frac{1}{\alpha+\beta}\left(\beta + \alpha(1-\alpha-\beta)^{10}\right) = 0.167} \quad \left(\simeq \tfrac{1}{6}\right).$$

7a) State 2 can be reached after 2, 4, 6, ... steps.
Period = gcd$\{2,4,6,\dots\}$ $\underline{\underline{= 2.}}$

b) $\underset{\sim}{m}_A$ is the minimal non-negative solution to the equations:

$$m_{1A} = m_{5A} = 0$$

$$m_{iA} = 1 + \sum_{j=1}^{5} p_{ij}\, m_{jA} \quad \text{for } i = 2, 3, 4.$$

So $m_{2A} = 1 + \frac{3}{5} * 0 + \frac{2}{5} m_{3A} \Rightarrow m_{2A} = 1 + \frac{2}{5} m_{3A}$ (*)

Also $m_{3A} = 1 + \frac{3}{5} m_{2A} + \frac{2}{5} m_{4A}$

$$= 1 + \tfrac{3}{5}\left(1 + \tfrac{2}{5} m_{3A}\right) + \tfrac{2}{5} m_{4A} \quad \text{from (*)}$$

$$\frac{19}{25} m_{3A} = \frac{8}{5} + \frac{2}{5} m_{4A} \quad (**)$$

Also $m_{4A} = 1 + \frac{3}{5} m_{3A} + \frac{2}{5} * 0$

$$= 1 + \frac{3}{5}\cdot\frac{25}{19}\cdot\left(\frac{8}{5} + \frac{2}{5} m_{4A}\right)$$

$$\frac{13}{19} m_{4A} = \frac{43}{19}$$

$$\Rightarrow \underline{\underline{m_{4A} = \frac{43}{13} \quad (3.31 \text{ steps}).}}$$

Thus $m_{3A} = \frac{25}{19}\left(\frac{8}{5} + \frac{2}{5}\left(\frac{43}{13}\right)\right) = \frac{50}{13}$ from (**)

and $m_{2A} = \frac{33}{13}$ from (*).

So $\underline{\underline{\underset{\sim}{m}_A = \left(0, \frac{33}{13}, \frac{50}{13}, \frac{43}{13}, 0\right).}}$

8) a) $G_{ij}(s) = \mathbb{E}(s^{T_{ij}})$.

<u>Case (i) : $j > i$.</u> Then $T_{ij} = T_{i,i+1} + T_{i+1,i+2} + \dots + T_{j-1,j}$:

the sum of $(j-i)$ independent random variables, all with the same distribution as T_{01}.

So $G_{ij}(s) = \mathbb{E}(s^{T_{ij}}) = \mathbb{E}\left(s^{T_{01}^{(1)} + \dots + T_{01}^{(j-i)}}\right)$

$= \left\{\mathbb{E}(s^{T_{01}})\right\}^{j-i}$ by independence

$$\underline{\underline{G_{ij}(s) = \{H(s)\}^{j-i} \quad \text{when } j > i}}$$

<u>Case (ii) : $j < i$.</u> Then $T_{ij} = T_{i,i-1} + T_{i-1,i-2} + \dots + T_{j+1,j}$:

the sum of $(i-j)$ independent r.v.s, each with the same distn as T_{10}.

But $T_{10} \sim T_{01}$ by symmetry : $\mathbb{P}(\text{take step up}) = \mathbb{P}\left(\begin{smallmatrix}\text{take}\\ \text{step}\\ \text{down}\end{smallmatrix}\right)$.

So $\underline{\underline{G_{ij}(s) = \{H(s)\}^{i-j} \quad \text{when } i > j.}}$

<u>Case (iii) : $j = i$.</u> Then $T_{ii} = \begin{cases} 1 + T_{i+1,i} & \text{with prob. } \frac{1}{2} \\ 1 + T_{i-1,i} & \text{with prob. } \frac{1}{2} \end{cases}$

(by conditioning on the first step).

So $G_{ii}(s) = \mathbb{E}(s^{T_{ii}}) = \frac{1}{2}\mathbb{E}(s^{1+T_{i+1,i}}) + \frac{1}{2}\mathbb{E}(s^{1+T_{i-1,i}})$

$= \frac{1}{2}s\,\mathbb{E}(s^{T_{10}}) + \frac{1}{2}s\,\mathbb{E}(s^{T_{01}})$

$= \frac{1}{2}s\,H(s) + \frac{1}{2}s\,H(s)$.

So $\underline{\underline{G_{ii}(s) = s\,H(s)}}$.

8b) $H(s) = \mathbb{E}(s^{T_{01}})$.

Condition on the first step taken:

$$T_{01} = \begin{cases} 1 & \text{if first step moves to } 1 \quad (\text{prob} = \tfrac{1}{2}) \\ 1 + T_{-1,1} & \text{if first step moves to } -1. \quad (\text{prob} = \tfrac{1}{2}) \end{cases}$$

Thus $\mathbb{E}(s^{T_{01}}) = \frac{1}{2}\mathbb{E}(s^1) + \frac{1}{2}\mathbb{E}(s^{1+T_{-1,1}})$

$= \frac{1}{2}s + \frac{1}{2}s\,\mathbb{E}(s^{T_{-1,1}})$

$= \frac{1}{2}s\,(1 + G_{-1,1}(s))$ by definition

$= \frac{1}{2}s\,(1 + \{H(s)\}^2)$ from part (a).

Thus $H(s) = \frac{1}{2}s + \frac{1}{2}s\,(H(s))^2$

$\Rightarrow \quad s\,H(s)^2 - 2H(s) + s = 0$

$\Rightarrow \quad H(s) = \dfrac{2 \pm \sqrt{4 - 4s^2}}{2s}$

$$\underline{H(s) = \frac{1 \pm \sqrt{1-s^2}}{s}}$$

Need to determine whether to use the + root or the − root :

consider $H(0) = \mathbb{P}(T_{01} = 0) = 0$, because it is not possible to reach state 1 from state 0 in 0 steps.

+ root $\Rightarrow H(0) = \dfrac{1+1}{0} = \infty$ $\times$ not possible

− root $\Rightarrow H(0) = \dfrac{1-1}{0}$ which can converge to 0 in the limit (prove by L'Hospital's Rule but not necessary here).

Thus the − root is required, so $\underline{\underline{H(s) = \dfrac{1-\sqrt{1-s^2}}{s}}}$ as stated.

The radius of convergence is $\underline{\underline{s = 1}}$ (it is always ≥ 1 for a PGF, and here $H(s) \notin \mathbb{R}$ if $s > 1$.)