

(a) Define the following expectations:

$$m_L = \mathbb{E}(\text{number of balls before the end} \mid \text{start just before entering box L})$$

$$m_R = \mathbb{E}(\text{number of balls before the end} \mid \text{start just before entering box R})$$

Using first-step analysis, the equations are:

$$m_L = 1 + \frac{1}{6}m_L + \frac{3}{6}m_R \quad (\text{a})$$

$$m_R = 1 + \frac{2}{4}m_L + \frac{1}{4}m_R \quad (\text{b})$$

Rearranging (b):

$$\frac{3}{4}m_R = 1 + \frac{2}{4}m_L$$

$$\Rightarrow m_R = \frac{4}{3} + \frac{2}{3}m_L. \quad (\text{c})$$

Substituting (c) in (a):

$$\frac{5}{6}m_L = 1 + \frac{2}{3} + \frac{1}{3}m_L$$

$$\frac{3}{6}m_L = \frac{10}{6}$$

$$m_L = \frac{10}{3}$$

So also, from (c):

$$m_R = \frac{32}{9}$$

(5)

(b) From the diagram,

$$\mathbb{E}(N) = \frac{1}{2}m_L + \frac{1}{2}m_R = \frac{1}{2} \left\{ \frac{10}{3} + \frac{32}{9} \right\} = \frac{31}{9}.$$

(1)

(c)

$$\mathbb{P}(\text{L, R, R, Stop}) = \frac{1}{2} \times \frac{3}{6} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64} \quad (0.0156).$$

(2)

(d) Define the following probabilities:

$$p_S = \mathbb{P}(\text{get no Lefts before the end} \mid \text{start at state Start})$$

$$p_R = \mathbb{P}(\text{get no Lefts before the end} \mid \text{start at state R})$$

First-step analysis:  $p_S = \frac{1}{2}p_R + \frac{1}{2} \times 0$

$$p_R = \frac{1}{4}p_R + \frac{1}{4} \times 1 + \frac{2}{4} \times 0$$

$$\Rightarrow \frac{3}{4}p_R = \frac{1}{4}$$

$$p_R = \frac{1}{3}$$

$$\therefore p_S = \frac{1}{6}.$$

The required probability is  $p_s = \frac{1}{6}$ .

(3)

- (e) We need to define a new state,  $L_2$ , that describes getting the second Left in a row. Let  $L_1$  be a state describing the first Left in any sequence of Lefts. We can terminate the diagram at  $L_2$  because our interest is only in whether we hit state  $L_2$  before we hit state **Stop**: we don't care what happens after  $L_2$ .

The new diagram is here.

Define the following probabilities:

$$h_S = \mathbb{P}(\text{hit state } L_2 \mid \text{start at Start})$$

$$h_L = \mathbb{P}(\text{hit state } L_2 \mid \text{start at } L_1)$$

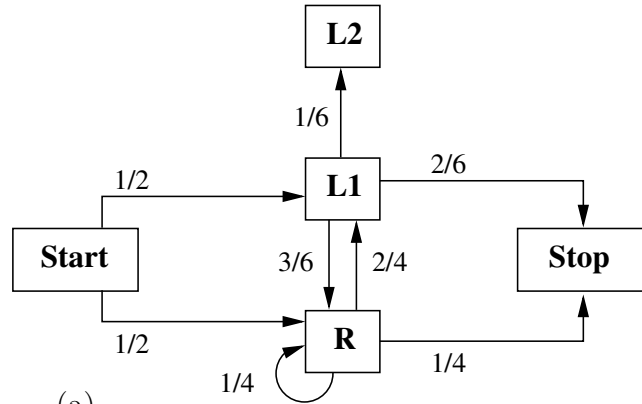
$$h_R = \mathbb{P}(\text{hit state } L_2 \mid \text{start at } R)$$

First-step analysis:

$$h_S = \frac{1}{2}h_L + \frac{1}{2}h_R \quad (a)$$

$$h_L = \frac{1}{6} + \frac{3}{6}h_R \quad (b)$$

$$h_R = \frac{2}{4}h_L + \frac{1}{4}h_R \quad (c)$$



$$(c) \Rightarrow h_R = \frac{2}{3}h_L$$

$$\text{Subst in (b)} \Rightarrow h_L = \frac{1}{6} + \frac{1}{3}h_L$$

$$\Rightarrow h_L = \frac{1}{4}$$

$$\Rightarrow h_R = \frac{1}{6}$$

Thus the probability of ever getting two lefts in a row is  $h_S = \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{6} = \frac{5}{24}$  (0.208).

(6)

- (f) **False:** The new diagram can **not** give the same distribution of  $N$  as the first diagram, for any  $p$  and  $q$ . The reason is that Left and Right balls do need to be distinguished. The probability of stopping after a Left ball ( $2/6$ ) is different from the probability of stopping after a Right ball ( $1/4$ ), so the two states can not be treated as having an identical stopping probability  $q$ .

To be sure that there is no  $p$  and  $q$  that can give the same distribution of  $N$ , we can show that we would need  $q = \frac{1}{\mathbb{E}(N)} = \frac{9}{31}$ ,  $p = \frac{22}{31}$  to match the value of  $\mathbb{E}(N)$  given in (b). However, with these values of  $p$  and  $q$ , we would get  $\mathbb{P}(N = 1) = \frac{9}{31}$  with the new diagram, but  $\mathbb{P}(N = 1) = \frac{1}{2} \left( \frac{2}{6} + \frac{1}{4} \right) = \frac{7}{24}$  for the old diagram. These are not the same, so no values of  $p$  and  $q$  can give the required distribution of  $N$ .

(3)

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**Total: (20)**