CHAPTER 13

of *Chance Encounters* by C.J.Wild and G.A.F. Seber

Control Charts

This chapter discusses a set of methods for monitoring process characteristics over time called *control charts* and places these tools in the wider perspective of quality improvement. The time series chapter, Chapter 14, deals more generally with changes in a variable over time. Control charts deal with a very specialized type of problem which we introduce in the first subsection. The discussion draws on the ideas about Normally distributed data and about variability from Chapter 6, about the sampling distributions of means and proportions from Chapter 7, hypothesis tests from Chapter 9 and about plotting techniques from Chapters 2 and 3.

13.1 Introduction

13.1.1 The Setting

The data given in Table 13.1.1, from Gunter [1988], was produced by a process that was manufacturing integrated circuits (ICs). The observations are coded measures of the thickness of the resistance layer on the IC for successive ICs produced. The design of the product specifies a particular thickness, here 205 units. Thus, 205 is the *target value*. If the thickness of the layer strays too far from 205 the performance of the IC will be degraded in various ways. Other manufacturers who buy the ICs to incorporate in their own products may well impose some limits on the range of thicknesses that they will accept. Such limits are called *specification limits*. Any ICs that fall outside this range are unacceptable to these customers.

So the manufacturer is trying to manufacture ICs with the target resistance-layer thickness of 205, but despite the company's best efforts, the actual thicknesses vary appreciably. This is typical of the products of any process. No two products are ever absolutely identical. There are differences due to variation in raw materials, environmental changes (e.g. humidity and temperature), variations in the way machines are operating, variations in the way that people do

things. In addition to variation in the actual units themselves, the measurement process introduces additional variation into our data on the process. As discussed in Section 6.4.1, a general principle of producing good quality products (and services) is that variability must be kept small. Thus, in the above example, we want to produce ICs with a resistance-layer thickness close to the target value of 205 and varying as little as possible.

Table 13.1.1 : Coded Thickness of Resistance Layer
on Integrated Circuit[a]

206	204	206	205	214	215	205	202	210	212	202	208	207	218
188	210	209	208	203	204	200	198	200	195	201	205	205	204
203	205	202	208	203	204	208	215	202	218	220	210		

[a]Source: Gunter [1988].   Data in time order (1st, 2nd, 3rd, ...) reading across rows.
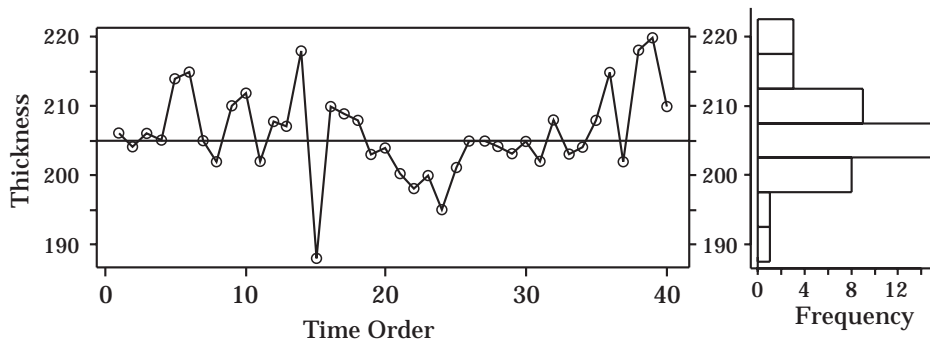

Figure 13.1.1 : Run chart and histogram of the data in Table 13.1.1.

Run chart

The data in Table 13.1.1 is plotted in two different ways in Fig. 13.1.1. The left-hand side of Fig. 13.1.1 is a *run chart*, namely a scatter plot of the measurements versus the time order in which the objects were produced ($1=1st$, $2=2nd$, etc.). The data points are linked by lines, a practice which enables us to see patterns that are otherwise not visible. Run charts provide a useful way of looking at the data to see if (and how) things have been changing over time. We add a horizontal line at the position of the target value if we want to see whether the process is centered on target or straying from target. Alternatively, we could use a horizontal (center-) line at the position of the mean if we wanted a visual basis for detecting changes in average level over time.

The right-hand side of Fig. 13.1.1 gives a histogram of the IC thicknesses which highlights the average level and variability of the whole set of measurements.
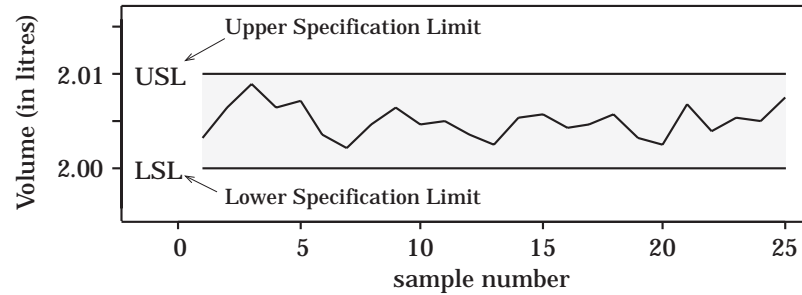
Figure 13.1.2 :   The volumes of 25 successive cartons
sampled at 5-min. intervals.

Specification limits

Fig. 13.1.2 charts the volumes of the contents of 2-liter cartons of milk sampled from a production line every 5 minutes. Fig. 13.1.2 has introduced a new feature, so-called *specification limits*. These are *USL* (Upper Specification Limit) and *LSL* (Lower Specification Limit). Such limits externally imposed (e.g. imposed by the customer). Units falling outside these limits are unacceptable and will be rejected.[1] It is the manufacturer's job to come up with a process that produces units that fall within the specification limits. The milk-carton process is doing a good job of delivering product that falls within the specification limits. Fig. 13.1.3 shows 3 processes that are failing to deliver within specifications for various reasons: the mean level at which process (a) is producing is too high and would have to be brought down to meet specifications; process (b) is too variable; and process (c) is too variable and the mean level is too low.
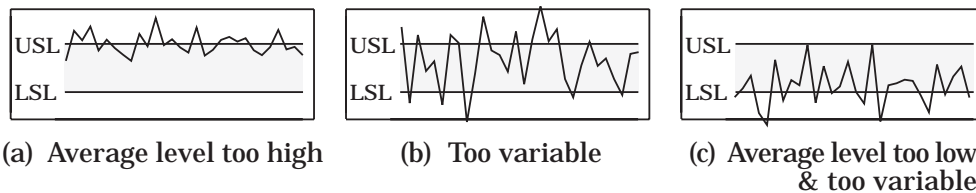

(a) **Average level too high**   (b) **Too variable**   (c) **Average level too low & too variable**

Figure 13.1.3 :   Three processes that are failing to meet specifications.

The above ideas apply to more than just manufacturing processes. We can use run charts and control charts to monitor waiting times for bank customers, numbers of complaints, error rates in handling insurance claims or ticketing airline passengers, customer satisfaction ratings, delivery times, and so on.

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[1]Specification limits, which apply only to individual units only, are very different from *control limits* (to follow) which indicate the level of variability expected in a process from its past history and often apply to averages. Charts showing *control* limits and charts showing *specification* limits can look very similar so look carefully at the fine print.

13.1.2 Statistical stability

A process is *statistically stable over time* (with respect to characteristic X) if the distribution of X does not change over time – see Fig. 13.1.4(a). You may wish to think of this in terms of stem-and-leaf plots constructed from data collected over separate time intervals (e.g. from different days) being very similar. Stability enables us to predict the range of variability to expect in our product in the future.[2] We can then try to develop systems that can cope with this level of variability. With an unstable process like that in Fig. 13.1.4(b), we have no idea what to expect, making intelligent forward-planning virtually impossible.
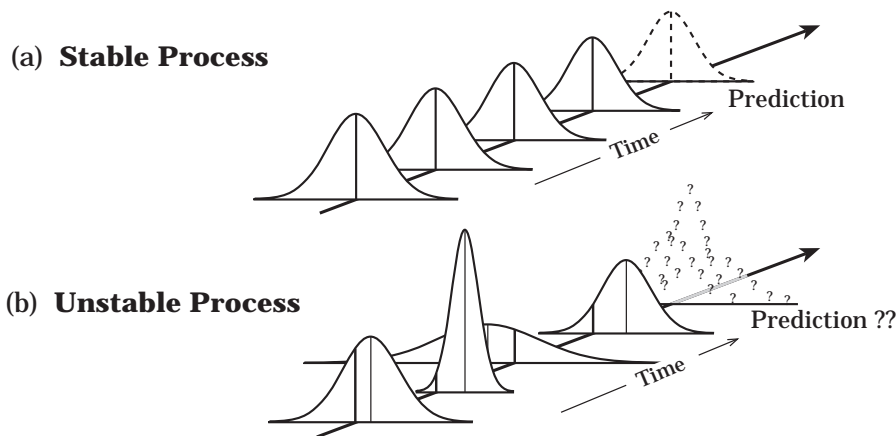
Figure 13.1.4 :   Consecutive distributions of X.

[Modeled on a figure in *Process Capability and Continuous Improvement*, Ford Motor Company.]

When plotted on a run chart, a *statistically stable* process bounces about its center in a random fashion. Non-random behavior tells us that the process is not statistically stable over time. The IC process plotted in Fig. 13.1.1 is not stable over time as we see a definite upwards trend in the latter half of the plot. However, the process in Fig 13.1.2 and all those depicted in Fig. 13.1.5 are stable. How do we know? In Fig. 13.1.5, all observations were sampled from the same Normal distribution using a computer.

Figure 13.1.5 :   Three stable processes.

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[2]This differs from other prediction problems you saw in regression where we were trying to predict individual values.

Gross *outliers* are a sign of an unstable process. The graphs in Fig. 13.1.6 show processes that are unstable in other ways. All are plotted using computer-generated random numbers from a Normal distribution. In Figs 13.1.6(a) and (b) the standard deviation was always the same. Halfway along the sequence in Fig. 13.1.6(a), we began to increase the mean of the distribution sampled from by a small amount each time an observation was taken. This is an exaggeration of what might happen if, for example, there was gradual wear in one of the machine components. In Fig. 13.1.6(b) there is an abrupt change of mean half way along the sequence.[3] This could arise, for example, because one component has partial failure or there is a change in operator who does not follow the procedure properly. Fig. 13.1.6(c), shows increased variability in the second half of the sequence. The same mean level was used throughout, but one third of the way along the sequence, we began to increase the standard deviation used. Machine wear could be a possible cause of behavior like that visible in Fig. 13.1.6(c). There are many other ways in which a process can be unstable.

In the quality literature, a process is said to be ***in control*** with respect to the characteristic[4] X if the distribution of X appears to be statistically stable over time. This name is rather unfortunate as "the process is in control" does ***not*** mean that we are actually making the process behave as we wish. Just as a stable process is "in control", an unstable process is ***out of control***, as in Fig. 13.1.6.

| (a) Mean level drifting | (b) Sharp change | (c) Variability increasing |
| upwards | in mean level | |
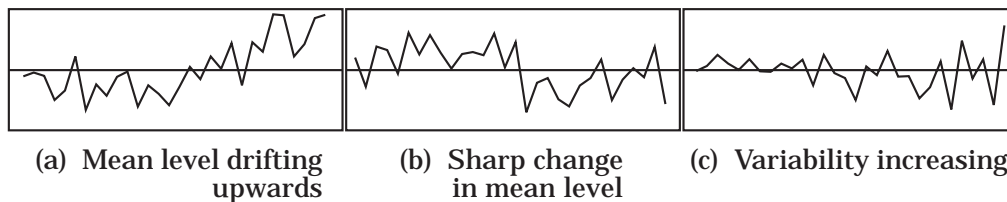
Figure 13.1.6 :   Three unstable (out-of-control) processes.

13.1.3 The function of control charts

The run chart provides a picture of the history of the performance of the process. Control charts will place additional information onto the run chart – information aimed at helping us to decide how to react, right now, in response to the most recent information about the process shown in the charts. They tell us about things we should do, and also about things we should refrain from doing.

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[3]The first half of the sequence was generated from Normal($\mu = 2.01, \sigma = 0.01$) and the second half from Normal($\mu = 2.00, \sigma = 0.01$), cf. the level and variability in the milk example.
[4]A process can be in control with respect to some characteristics (variables) while others are out of control.

The temptation to tamper

Let us go back to the thicknesses of IC resistance layers in Fig. 13.1.1. We know that the target for the thickness is 205 and we want to have as little variation about this target as possible. Faced with a variable output from the process, a natural human response is to tinker with the system. The resistance layer on this IC is a little thick so let's change some settings to try and make the next IC-layer thinner. If the next IC-layer is still too thick make an even larger adjustment, and an even larger one. If we get one IC where the layer is too thin, make an adjustment in the other direction and so on. One of the major discoveries of Walter Shewhart, who invented the first control charts in the late 1920's, was that when a process is subject only to variation which looks random, such *tampering* with a process only makes things worse (more variable). In such situations, one should keep one's hands off the process until the causes of the variation are well understood. Ever since, control charts have been helping establish this as common practice in industry.

Looking locally for causes

Another natural human impulse we often have to guard against is the tendency to look around very locally for the causes of a problem. The authors' country, New Zealand, is a very small country in population terms. Because it is so small, there is a great deal of variability between the numbers of people killed on the roads in a holiday weekend one year and the number on the corresponding holiday the next year. At the end of every holiday weekend, a police official will appear on the television news to explain why the figures turned out the way they did this year – to explain to us what we have (collectively) done right or wrong this time. They "look locally" for causes. In other words, their explanations arise from asking themselves questions like, "What have we changed recently?" or "What was unusual this time?" It turns out that there are times when "looking locally" is a good strategy for finding causes of variation and even more times when it is a bad strategy. Control charts help us tell these situations apart.

Common-cause versus special-cause variation

When trying to understand the variation evident in a run chart, it is useful to begin with the idea of stable, background variation which appears random and is called **common-cause** variation. (Further variation may be superimposed on top of this.) Common-cause variation is present to some extent in all processes. It is an inherent characteristic of the process which stems from the natural variability in inputs to the process and its operating conditions. When only common-cause variation is present, adjusting the process in response to each deviation from target **in**creases the variability. "Looking locally" for causes of common-cause variation is fruitless. This variability is an inherent characteristic of the way the system operates and can only reduced by changing the system itself in some fundamental way.

On the other hand, for variation which shows up as outliers or specific identifiable patterns in the data, asking questions like, "What have we changed recently?" or "Did something particularly unusual happen just prior to this and what was it?" very often turns up a real cause, such as an inadequately trained operator or wear in a machine. This latter type of variation is classified as *special-cause* variation.[5] Special-cause variation is unusual variation, so it makes sense to look for "something unusual" as its cause. The investigation should take place as soon as possible after the signal has been given by the chart so that memories of surrounding circumstances are still fresh. If the cause can be located and prevented from recurring, a real improvement to the process has been made. In summary, control charts tell us when we have a problem that is likely to be solved by looking for "something unusual" as its cause.

Reducing common-cause variation

The reduction of common-cause variation is also very important, but control charts are not designed for this task. Quite different tools and ways of thinking are required. For example, it may be possible and worth while to control variability in some inputs to the process or aspects of the operating environment. In order to do this, however, we must first find out what types of variability in the inputs and operating environment are most important as causes of variability in the end-product. It may be possible to modify some internal parts of the process. Making changes without knowing what effects they are likely to have on the product constitutes tampering, with all its ill effects. Making *informed* changes requires planned investigation. Methods include observing the effects of experimental interventions, and also performing observational studies which relate "upstream" variables (such as measurements on aspects of incoming raw materials, operators, procedures, machines involved, etc.) to characteristics of the product. Regression methods are often useful for this. But let us return to control charts.

Control chart construction – the basic idea

If we start with a process showing a stable pattern of variation, control charts signal a change from that pattern — when things have started to "go wrong." They try, informally, to trade off two sets of costs. We want the signal to come early enough to avoid accumulating big costs from low-quality production, but we do not want to react to common-cause variation. So we need some criteria for deciding whether what we are seeing is only background variation or whether the process is starting to go out of control.

> Normal distribution: **99.7%** of observations fall within $\mu \pm 3\sigma$
> (3-sigma limits)

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[5] Also called **assignable cause**.

Recall from Section 6.2.1 that 99.7% of observations sampled from a Normal distribution fall within 3 standard deviations of the mean, i.e. between the limits $\mu \pm 3\sigma$. Thus, if observations started to appear outside these limits then we would suspect that the process is no longer in control, and that the distribution of X had changed.[6] What if the distribution of X is not Normal? A famous result called the Chebyshev's inequality tells us that, *irrespective* of the distribution, at least 89% of observations fall within the 3-sigma limits.[7] The probability of falling within the 3-sigma limits increases towards 99.7% as the distribution becomes more and more Normal. Many physical measurements are approximately Normally distributed and we shall improve the approximation by using means of groups of observations (see Section 13.2.1). Thus, 3-sigma limits give a simple and appropriate way of deciding whether or not a process is in control. You might ask why not use narrower limits such as 2-sigma limits. Experience has shown that when 2-sigma limits are used, the control chart often indicates special causes of variation that cannot be found. When 3-sigma limits are used, a diligent search will often unearth the special cause.[8]

We will learn about three kinds of control charts: $\overline{x}$-charts which are used for looking for a change in the average level; R-charts to look for changes in variability; and $\widehat{p}$-charts which monitor proportions (e.g. proportion of items which are defective).[9] For most of the chapter, we will concentrate on points falling outside the 3-sigma limits as a signaling the likely presence of a special cause. Section 13.6 will introduce a range of patterns to look for which are also signals of the presence of special causes.

Quiz for Section 13.1

1. What is a run chart?
2. What are the main purposes of control charts?
3. Two types of variation were described. What are they?
4. What type of variation are control charts intended to detect? What do we do when we detect evidence of such variation?
5. When all variation is common-cause variation, there is something we should not do with the process. What is it and why?
6. How should we approach the reduction of common-cause variation?
7. What is meant by a process being "in control"? What does "in control" not mean?

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[6]Thus, control charts can be thought of as visual hypothesis tests.

[7]Chebyshev's inequality: $\mathrm{pr}(|X - \mu| < k\sigma) > 1 - (1/k)^2 \approx 0.89$ when $k = 3$. (It applies to distributions that have a finite standard deviation.)

[8]The above is an over-simplification. One has to trade off false-positive rates (expending time and effort looking and not finding anything) and false-negative rates (doing nothing when one should have acted) in the environment in which one is working. It may also depend on the frequency with which data is generated. Less stringent 2-sigma limits are often used for processes that generate data only very slowly (e.g. monthly accounts). Some processes generate huge numbers of data points each day, and sometimes even 4-sigma limits are used for these.

[9]$\widehat{p}$-charts are conventionally called p-charts. The change from p to $\widehat{p}$ emphasizes the fact that *sample* proportions are charted.

8. Describe one advantage of having a stable process over having an unstable process.
9. Why do we use 3-sigma limits and not 2-sigma limits?
10. What percentage of observations on a stable, Normally-distributed process fall within 3-sigma limits?

13.2   Control Charts for Groups of Data

13.2.1   Monitoring average level: the $\bar{x}$–chart

The $\bar{x}$-chart is used to look for changes in the average value of X-measurements as time goes on. As the measured characteristic of the process may not be Normally distributed, we make use of the Central Limit effect by working with sample means instead of individual X-values so that we are working with quantities that have a distribution that is closer to Normal. Typically data are collected in *at least* 20 subgroups of size 3 to 6 (typically 5) measurements and the mean of each of subgroup is computed. Recall from Section 7.2 that if we are sampling from a distribution with mean μ and standard deviation σ, the sample means from subgroups of n observations vary according to a distribution with mean and standard deviation given by:

$$\text{(Subgroups of size } n) \qquad \mu_{\bar{X}} = \mu, \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}.$$

and almost all subgroup means will lie within the 3-sigma limits:

(3-sigma limits for subgroup means) $\qquad \mu_{\bar{X}} \pm 3\,\sigma_{\bar{X}}.$

If we knew the true values of μ and σ, we would use $\mu_{\bar{X}} + 3\sigma_{\bar{X}}$ as the *upper control limit* (UCL), and $\mu_{\bar{X}} - 3\sigma_{\bar{X}}$ as the *lower control limit* (LCL). New subgroup means outside this range would be considered to provide a signal that the process was out of control. In practice however, one never knows μ and σ and the control limits must be estimated from the data.

As well-known American statistician J. Stuart Hunter likes to point out in his seminars, each time a new subgroup mean is plotted and one checks whether it is inside the control limits, one is essentially performing an hypothesis test graphically. The hypothesis is that the new mean comes from a distribution with mean $\mu_{\bar{X}}$. The alternative is that the mean has changed. Using $3\sigma_{\bar{X}}$ limits corresponds to rejecting the hypothesis only for *P-values* considerably smaller[10] than 5%. One reason for being very conservative in this sense is to protect against multiple-comparisons problems as so many tests are being performed (each new mean plotted corresponds to another test).

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[10]The minimum *P*-value for rejection would be 0.3% in the idealized situation in which the data was coming from a Normal distribution with known standard deviation.

Example 13.2.1 Telstar Appliance Company uses a process to paint refrigerators with a coat of enamel. During each shift, a sample of 5 refrigerators is selected (1.4 hours apart) and the thickness of the paint (in mm) is determined. If the enamel is too thin, it will not provide enough protection. If it's too thick, it will result in an uneven appearance with running and wasted paint. Table 13.2.1 below lists the measurements from 20 consecutive shifts. In the language of the previous paragraph, a sample of 5 from the same shift is a subgroup and we have 20 subgroups. Fig. 13.2.1 provides an $\overline{x}$-chart for this data. ∎

Construction of the $\overline{x}$-chart

Suppose we have nk observations made up of k subgroups each of size n. In Example 13.2.1, we have $k = 20$ subgroups of size $n = 5$.

The **center line** of the $\overline{x}$-chart (cf. Fig. 13.2.1) is plotted at the level of the sample mean (average) of the k subgroup means. This value[11] is denoted by $\overline{\overline{x}}$. This is a natural estimate of the true $\mu_{\overline{X}} = \mu$. In Example 13.2.1., $\overline{\overline{x}} = 2.514$ (see Table 13.2.1).

The control limits are estimates of $\mu_{\overline{X}} \pm 3\sigma_{\overline{X}}$. The **upper control limit** (UCL) is plotted at the level $\overline{\overline{x}} + 3\widehat{\sigma}_{\overline{X}}$ and the **lower control limit** (UCL) is plotted at $\overline{\overline{x}} - 3\widehat{\sigma}_{\overline{X}}$ where $\widehat{\sigma}_{\overline{X}}$ is an estimate of $\sigma_{\overline{X}} = \sigma/\sqrt{n}$.

To construct an estimate of $\sigma_{\overline{X}}$, we need an estimate of σ. The estimates, $\widehat{\sigma}_{\overline{X}}$, that are most often used in practice are given in Table 13.2.2. Estimate (i) is equivalent to estimating σ by $\overline{s}$ which is the sample mean of the k subgroup standard-deviations. In Table 13.2.1, we have 20 subgroups. Their standard deviations are given in the final column of the table and their average is $\overline{s} = 0.3101$. However, $\overline{s}$ is a biased estimate[12] of σ. The d_1 given in the formula is a correction factor chosen so that $\overline{s}/d_1$ is an unbiased estimate of σ when we are sampling from a Normal distribution. Values of d_1 are tabulated in Table 13.2.3.[13]

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[11]Note that for equal subgroup sizes, if we averaged all the nk individual observations, we would get the same value ($\overline{\overline{x}}$).

[12]Although s^2 values have population mean σ^2, s does not have mean σ.

[13]The most obvious candidate for an estimate of σ is s, the sample standard deviation of all nk individual observations. However, this value is not used. The average of the batch standard deviations is a measure of within-subgroup variability. The overall standard deviation also reflects between-subgroup variation. In practice, control charts often have to be set up in less than perfect circumstances and there may be special causes acting from one subgroup to the next during the set-up phase. It is the within-subgroup variability that one wants to estimate.

Table 13.2.1 :   The Thickness of Paint on Refrigerators
for Five Refrigerators from Each Shift

(Subgroup) Shift no.	Thickness (in mm)					Subgroup Mean	Range	Std Dev.
1	2.7	2.3	2.6	2.4	2.7	2.54	0.4	.1817
2	2.6	2.4	2.6	2.3	2.8	2.54	0.5	.1949
3	2.3	2.3	2.4	2.5	2.4	2.38	0.2	.0837
4	2.8	2.3	2.4	2.6	2.7	2.56	0.5	.2074
5	2.6	2.5	2.6	2.1	2.8	2.52	0.7	.2588
6	2.2	2.3	2.7	2.2	2.6	2.40	0.5	.2345
7	2.2	2.6	2.4	2.0	2.3	2.30	0.6	.2236
8	2.8	2.6	2.6	2.7	2.5	2.64	0.3	.1140
9	2.4	2.8	2.4	2.2	2.3	2.42	0.6	.2280
10	2.6	2.3	2.0	2.5	2.4	2.36	0.6	.2302
11	3.1	3.0	3.5	2.8	3.0	3.08	0.7	.2588
12	2.4	2.8	2.2	2.9	2.5	2.56	0.7	.2881
13	2.1	3.2	2.5	2.6	2.8	2.64	1.1	.4037
14	2.2	2.8	2.1	2.2	2.4	2.34	0.7	.2793
15	2.4	3.0	2.5	2.5	2.0	2.48	1.0	.3564
16	3.1	2.6	2.6	2.8	2.1	2.64	1.0	.3647
17	2.9	2.4	2.9	1.3	1.8	2.26	1.6	.7021
18	1.9	1.6	2.6	3.3	3.3	2.54	1.7	.7829
19	2.3	2.6	2.7	2.8	3.2	2.72	0.9	.3271
20	1.8	2.8	2.3	2.0	2.9	2.36	1.1	.4827
				Column mean =		2.514 $(\overline{\overline{x}})$	0.77 $(\overline{r})$	.3101 $(\overline{s})$

Table 13.2.2 :   Commonly Used Estimates of $\sigma_{\bar{X}}$

(i) $\widehat{\sigma}_{\bar{X}} = \dfrac{1}{d_1} \dfrac{\overline{s}}{\sqrt{n}},$ where $\overline{s}$ is the mean of the subgroup std dev's

(ii) $\widehat{\sigma}_{\bar{X}} = \dfrac{1}{d_2} \dfrac{\overline{r}}{\sqrt{n}},$ where $\overline{r}$ is the mean of the subgroup ranges

Values of d_1 and d_2 depend upon the subgroup size n.
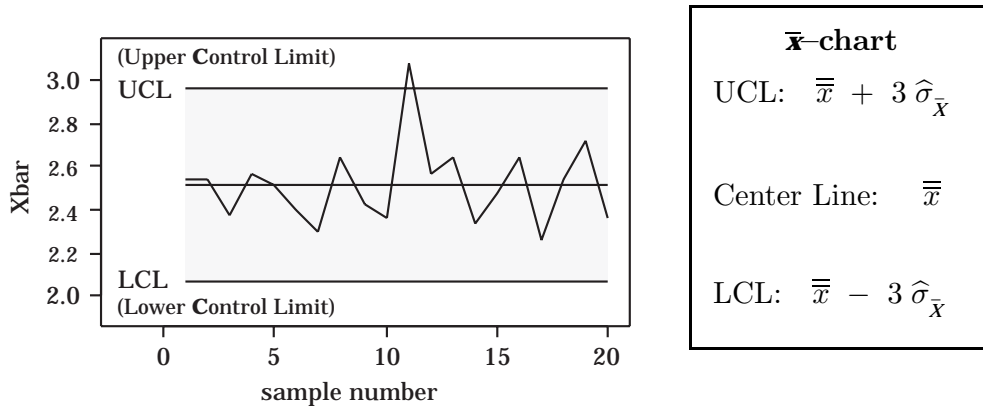A range of values are tabulated in Table 13.2.3.

x̄–chart

UCL: $\overline{\overline{x}} + 3\,\widehat{\sigma}_{\bar{X}}$

Center Line: $\overline{\overline{x}}$

LCL: $\overline{\overline{x}} - 3\,\widehat{\sigma}_{\bar{X}}$

Figure 13.2.1 : $\overline{x}$–chart for the data in Table 13.2.1.

The value of $\widehat{\sigma}_{\bar{X}}$ given as estimate (ii) in Table 13.2.2 corresponds to estimating σ using the information about spread contained in the average of the subgroup ranges, $\bar{r}$. It turns out that, for a Normal distribution, $\bar{r}/d_2$ is an unbiased estimate of σ. Values of d_2 are also tabulated in Table 13.2.3.

Because much less computation is required by method (ii) than method (i), method (ii) is usually used for charts constructed and updated by hand. It works very well with the small subgroup sizes used in practice. However, method (i) is better for computerized charts as $\bar{s}$ provides a more precise estimate of σ and is less affected by outliers.

Example 13.2.1 cont. For the data in Table 13.2.1, $\overline{\overline{x}} = 2.514$. The subgroup ranges are given as the second to last column of Table 13.2.1. The average of the subgroup ranges appears at the bottom of the column. Thus, we see that $\bar{r} = 0.77$. Since $n = 5$, Table 13.2.3 tells us that $d_2 = 2.3259$. Thus,

$$\widehat{\sigma}_{\bar{X}} = \frac{1}{d_2}\frac{\bar{r}}{\sqrt{n}} = \frac{1}{2.3259}\frac{0.77}{\sqrt{5}} = 0.14805,$$
$$UCL = \overline{\overline{x}} + 3\,\widehat{\sigma}_{\bar{X}} = 2.96,$$

and
$$LCL = \overline{\overline{x}} - 3\,\widehat{\sigma}_{\bar{X}} = 2.07.$$

The control chart is drawn in Fig. 13.2.1 and we see that we have a point (the 11th) outside the control limits indicating that the process is not in control. Subsequent points all look fine making this appear to be an isolated problem. Nevertheless, we would try to find out what caused the problem with the 13th observation and prevent the recurrence of such problems. ∎

Even having all of the sample means lie between the control limits is no guarantee that the process is control. There may still be some internal features of the chart which suggest instability and the presence of special causes. Some rules for detecting such problems are discussed later in Section 13.6. If there are points outside the control limits, then we should try and find special causes for these points. If causes are found then the offending points can be removed and the center line and control limits re-calculated.

Use of projected control limits

In Example 13.2.1, we used all of our data to construct the control limits. In practice, it is usual to have a set-up phase of perhaps 20 or 30 subgroups, from which the center line and control limits are calculated. If the process has been reasonably stable over the set-up phase, a chart will be constructed plotting the set-up data and the corresponding limits projected out into the future, as in Fig. 13.2.2. Future data points will be plotted on that chart to monitor the behavior of the process as time goes on. The limits will not be updated unless there has been a substantial change in the process.
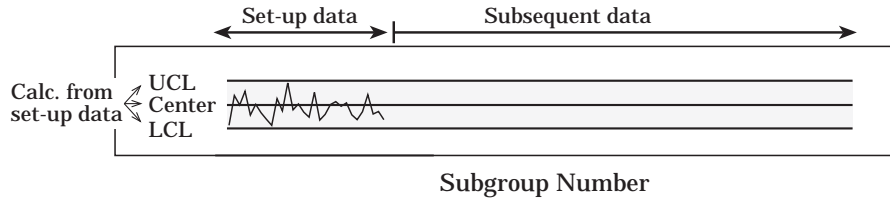
Figure 13.2.2 : Projected Control Limits.

Table 13.2.3 : Constants for the Construction of Control Charts[a]

n	d_1	d_2	$(\frac{3}{d_1\sqrt{n}})$ A_1^b	$(\frac{3}{d_2\sqrt{n}})$ A_2^b	D_3	D_4	B_3	B_4
2	1.7725	1.1284	1.1968	1.8800	0.0000	3.2665	0.0000	10.8276
3	1.3820	1.6926	1.2533	1.0233	0.0000	2.5746	0.0010	6.9078
4	1.2533	2.0588	1.1968	0.7286	0.0000	2.2820	0.0081	5.4221
5	1.1894	2.3259	1.1280	0.5768	0.0000	2.1145	0.0227	4.6167
6	1.1512	2.5344	1.0638	0.4832	0.0000	2.0038	0.0420	4.1030
7	1.1259	2.7044	1.0071	0.4193	0.0757	1.9243	0.0635	3.7430
8	1.1078	2.8472	0.9575	0.3725	0.1362	1.8638	0.0855	3.4746
9	1.0942	2.9700	0.9139	0.3367	0.1840	1.8160	0.1071	3.2656

[a]This table is also reproduced in Appendix A13 for quick reference.

[b]The A_i columns can be used to speed up the calculation of the $\bar{x}$-chart since,
using $\bar{s}$, $3\hat{\sigma}_{\bar{X}} = \frac{3}{d_1\sqrt{n}}\bar{s} = A_1\bar{s}$. Similarly, using $\bar{r}$, $3\hat{\sigma}_{\bar{X}} = A_2\bar{r}$.

Exercises on Section 13.2.1

1. A company produces dials for a machine. These dials are supposed to have a constant diameter. To check on the production process, the first 4 dials are selected every half hour for 12 hours giving a total of 96 observations. It was found that $\bar{\bar{x}} = 51.12\,\text{mm}$ and $\bar{r} = 0.46\,\text{mm}$. Find the upper and lower control limits.

2. Do you think that just looking for points which are outside the control limits provides enough evidence for the existence of an unstable process? Suppose someone tossed a coin ten times and you got an alternating sequence of heads and tails such as $HTHTHTHTHT$. Would you think that just chance is at work? (These ideas are explored further in Section 13.6.)

3. In Example 13.2.1, the out-of-control mean given for shift 11 was found to be due to an operator error. Recurrence has been prevented by giving the operator additional training. Delete this point from the data and recompute the center line and the control limits. Are your conclusions any different?

13.2.2 Control Charts for Variation: the R–chart

An $\bar{x}$–chart focuses attention on the constancy of average level (μ) and is not good at detecting changes in variability (σ).[14] An ***R-chart*** (or *range chart*) is specifically designed for detecting changes in variability. This time we plot the subgroup ranges, r_i, rather than the subgroup means. Fig. 13.2.3 is an R-chart for the refrigerator data in Table 13.2.1.

If μ_R and σ_R are respectively the mean and standard deviation of the range R of n observations sampled from a Normal distribution, then it can be shown that σ_R is of the form $\sigma_R = d\mu_R$ for some constant d. The desired upper control limit (UCL) for the subgroup ranges is $\mu_R + 3\sigma_R$. Now,

$$\mu_R + 3\sigma_R = \mu_R(1 + 3d) \quad \text{which is of the form} \quad D_4\mu_R.$$

Similarly, the lower control limit (LCL) is of the form $D_3\mu_R$. The formulae for UCL and LCL given on the right of Fig. 13.2.3 follow when we estimate μ_R by the sample mean of the subgroup ranges, namely $\bar{r}$. Values of D_3 and D_4 are tabulated in Table 13.2.3. Negative values of D_3 are not permitted as a sample range cannot be negative. Consequently, negative values are set to zero.[15]

In using the R–chart, subgroup ranges lying outside the control limits indicate that the process is "out of control". Trends in the R–chart may indicate a problem like wear in the machine.[16]
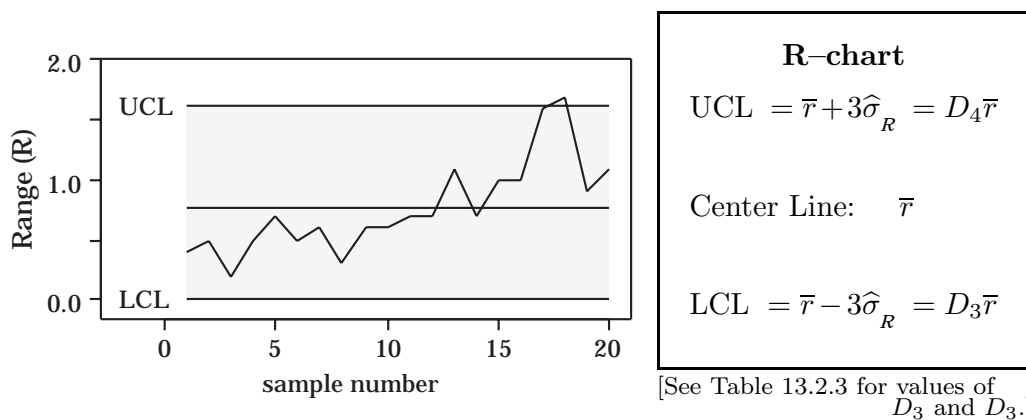

R–chart

UCL $= \bar{r} + 3\hat{\sigma}_R = D_4\bar{r}$

Center Line: $\bar{r}$

LCL $= \bar{r} - 3\hat{\sigma}_R = D_3\bar{r}$

[See Table 13.2.3 for values of D_3 and D_3.]

Figure 13.2.3 : R–chart for the data in Table 13.2.1.

Example 13.2.1 cont. We refer again to the refrigerator data in Table 13.2.1. From Table 13.2.1, $\bar{r} = 0.77$ and this forms the center line. Table 13.2.3 tells us that when $n = 5$, $D_3 = 0$ and $D_4 = 2.1145$. Thus we have,

$$UCL = D_4\bar{r} = 1.63 \quad \text{and} \quad LCL = D_3\bar{r} = 0.$$

---

[14]e.g. if μ is constant but σ occasionally becomes larger, then the process could still appear to be in control on an $\bar{x}$-chart because of the averaging effect.

[15]Note from Table 13.2.3 that $D_3 = 0$ for n less than 7.

[16]When $\bar{s}$ is readily calculated we can construct a so-called s–chart using $\bar{s}$ instead of $\bar{r}$ and constants B_3 and B_4 instead of D_3 and D_4 (see Gitlow et al. [1995, p. 246]).

The control chart, given in Fig. 13.2.3, clearly indicates that the process is out of control, and not just because one point is above UCL. The range seems to be steadily increasing, telling us that the variability of the original paint thickness is increasing over time. This picture is in strong contrast to the $\overline{x}$–chart of Fig. 13.2.1 which seems to suggest that things aren't too bad. Clearly an $\overline{x}$–chart is not sufficient on its own and needs to be supplemented with an R–chart. To sum up, the painting process that produced the data in Table 13.2.1 is out of control, not because of changes in mean thickness, but because of increasing variability in paint thicknesses.

Exercises on Section 13.2.2

1. Would you expect D_4 to increase or decrease with n, the subgroup size? Check your reasoning by referring to Table 13.2.3.

2. In the Exercises 13.2.1, problem 1, the subgroups of size 4 gave $\overline{r} = 0.46$ mm. Compute the control limits.

13.2.3   Control Charts for Proportions: $\widehat{p}$–chart

Example 13.2.2   Table 13.2.4 gives the number of units produced by a company each week during part of 1994 which require rework at certain stages of production because of faults. Rework is additional work required to bring a substandard unit up to standard and is an additional cost one wishes to avoid. We omit further details for reasons of confidentiality. The weeks are consecutive except for the last one (in 1995) when the factory closed down over the Christmas holiday period.[17]   We want to chart the percentages requiring rework over time to see whether the process is in control. Such a chart is given in Fig. 13.2.4. ∎

Example 13.2.2 is just one of many examples in which we want to chart the proportion of items produced by a process that are defective in some way. The resulting charts are traditionally called **p–charts**. However, they monitor the behavior of the *sample proportion* ($\widehat{p}$ in our notation) of defective items over time. We select subgroups of n items and compute the sample proportion ($\widehat{p}$) of defectives for each subgroup. We then plot each successive subgroup proportion on the chart, as in Fig. 13.2.4.

The limits $\mu_{\widehat{p}} \pm 3\sigma_{\widehat{p}}$ for $\widehat{p}$ are $p \pm 3\sqrt{\frac{p(1-p)}{n}}$. If the process is in control, then p, the unknown true probability that an item is produced defective, is the same for all items regardless of what subgroup they come from. If we take

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[17]Christmas is in the summer in NZ and many factories shut down for 2 or 3 weeks.

Table 13.2.4 :   The Number of Units per Week Requiring
Rework at Certain Stages of Production

Week ending (Day/month)	Subgroup No.	Requiring rework	Total production	Proportion $(\widehat{p})$
7/5	1	35	3662	.0096
14	2	52	3723	.0140
21	3	37	3633	.0102
28	4	31	3664	.0085
4/6	5	23	3448	.0067
11	6	31	2630	.0118
18	7	21	3580	.0059
25	8	30	3278	.0092
2/7	9	20	3797	.0053
9	10	20	3893	.0051
16	11	40	3991	.0100
23	12	65	3760	.0173
30	13	58	3590	.0162
6/8	14	78	3108	.0251
13	15	43	3759	.0114
20	16	30	3606	.0083
27	17	29	3530	.0082
3/9	18	56	3621	.0155
10	19	41	3888	.0105
17	20	32	3854	.0083
24	21	81	3864	.0210
1/10	22	74	3846	.0192
8	23	24	3856	.0062
15	24	42	4072	.0103
22	25	35	3693	.0095
29	26	15	3394	.0044
5/11	27	18	4157	.0043
12	28	25	4012	.0062
19	29	57	3698	.0154
26	30	57	3658	.0156
3/12	31	42	3236	.0130
10	32	71	3913	.0181
17	33	40	3655	.0109
24	34	24	3542	.0068
21/1	35	27	2356	.0115
	Totals:	1404	126967	

$\widehat{p}$–**chart**

$$\text{UCL} = \overline{p} + 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}$$

$$\text{Center Line:} \quad \overline{p}$$

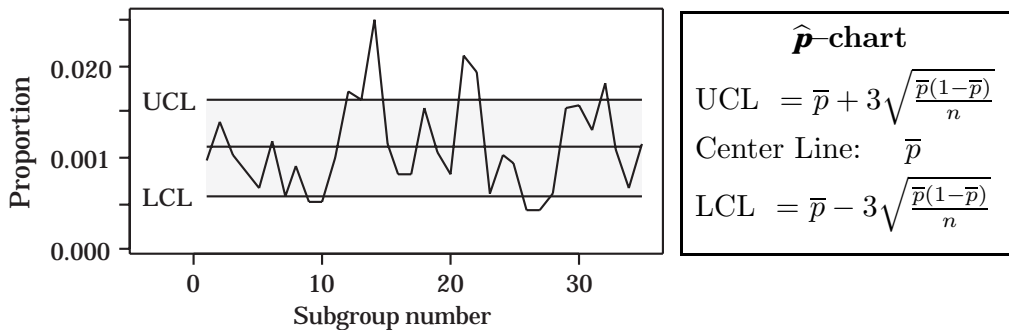$$\text{LCL} = \overline{p} - 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}}$$

Figure 13.2.4 : $\widehat{p}$–chart for the data in Table 13.2.4.

k (e.g. at least 20) subgroups of data all of the same size n so that the total number inspected is nk, then a natural estimate of p is the average of all of the subgroup proportions,

$$\bar{p} = \frac{1}{k}(\hat{p}_1 + \hat{p}_2 + \cdots + \hat{p}_k)$$
$$= \frac{1}{nk}(n\hat{p}_1 + n\hat{p}_2 + \cdots + n\hat{p}_k)$$
$$= \frac{\text{Total number of defectives}}{\text{Total number inspected}}.$$

The estimate $\bar{p}$ leads to the control limits given to the right of Fig. 13.2.4. The latter expression for $\bar{p}$ can be used even for unequal subgroup sizes. We note that if LCL falls below zero[18] we use $LCL = 0$.

How large should the subgroup size be? In order for the Central Limit effect to begin to operate, n should be such that[19] $np \geq 2$, which often means that $n > 50$. Since n is usually large, more data is needed for a p–chart than for $\bar{x}$– and R–charts.

Unfortunately, the subgroup size n is not constant in Example 13.2.2. The number of items produced in week i, denoted n_i, varies somewhat from week to week. So what value of n do we use to calculate the control lines? The most accurate method would be to work out individual values of UCL and LCL using n_i. However, this ignores the motivational impact of a simple easy-to-understand chart. An alternative approximation is to use $\bar{n}$, the *average value* of n, in calculating UCL and LCL. When will this approximation be satisfactory? One rule of thumb is to use it when n does not vary from $\bar{n}$ by more than about 25% of $\bar{n}$. Once the control chart has been plotted, any point i near or outside the limits could be checked by recalculating the limits using n_i.

Example 13.2.2 cont.   Using the totals in Table 13.2.4, the center line for our $\hat{p}$–chart is plotted at

(Center Line)
$$\bar{p} = \frac{\text{Total number of defectives}}{\text{Total number inspected}} = \frac{1404}{126967} = 0.011058.$$
$$\text{Also,} \bar{n} = \frac{\sum n_i}{k} = \frac{\text{Total production}}{\text{Number of weeks observed}} = \frac{126967}{35} = 3627.63.$$

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[18]We then only need to check whether UCL is exceeded or not as we can never fall below LCL, though we may get a $\hat{p}$ equal to zero when there are no defects in a subgroup.

[19]A variety of rules are suggested in the literature e.g. Gitlow et al. [1995] suggest using $np \geq 2$. This implies that we should expect to find at least 2 defectives in a subgroup. Our previous 10% rule for approximate Normality of the underlying Binomial distribution is too conservative for control chart applications.

A variation of 25% on this would give a range of n_i-values of about 2700 to 4500 which covers all the values of n except two, namely subgroups 6 and 35 (which were based on shorter weeks because of public holidays). We will therefore use the mean value of n, namely $\overline{n}$, to calculate the control limits. Hence, using $\overline{p} = 0.011058$ and $\overline{n} = 3627.63$, we have

$$UCL(\widehat{p}) \; = \; \overline{p} + 3\sqrt{\frac{\overline{p}(1 - \overline{p})}{\overline{n}}} \; = \; 0.0163,$$

and
$$UCL(\widehat{p}) \; = \; \overline{p} - 3\sqrt{\frac{\overline{p}(1 - \overline{p})}{\overline{n}}} \; = \; 0.0058.$$

The resulting $\widehat{p}$–chart is given in Fig. 13.2.4. We see that the process is out of control. Since n varies we could look at each point outside the limits and recompute UCL and LCL using the correct value of n_i. For example, for subgroup 12, $n_{12} = 3760$ and

$$UCL(\widehat{p}_{12}) = 0.011058 + 3\sqrt{\frac{0.011058(1 - 0.011058)}{3760}} = 0.0162,$$

which is slightly less than[20] $UCL(\overline{p})$.

   This example has used historical data to illustrate the calculations and looked for evidence of out-of-control behavior in the history. In practice, historical data is used to set up limits for charting the subsequent behavior of the process and action is taken as soon as out-of-control behavior is seen.

Table 13.2.5 :   Forestry Data

Month	No. defect.	Total logs	% defect.	Month	No. defect.	Total logs	%. defect.
Nov/93	224	5708	3.92	Aug/94	253	4064	6.23
Dec	181	3553	5.09	Sep	172	3577	4.81
Jan/94	187	3293	5.68	Oct	172	3711	4.63
Feb	122	2808	4.34	Nov	248	3944	6.29
Mar	134	2924	4.58	Dec	171	2925	5.85
Apr	235	3472	6.77	Jan/95	261	4013	6.50
May	151	2782	5.43	Feb	263	4944	5.32
Jun	227	4051	5.60	Mar	184	3356	5.48
Jul	226	4493	5.03	Apr	174	3835	4.54

Exercise on Section 13.2.3

   The data in Table 13.2.5 relate to sampling audits carried out on a monthly basis on samples of logs produced by a forestry operation. Each month,

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[20]In fact for points that appear out of control, we only need to recompute the limits for those points in which $n_i < \overline{n}$. When $n_i > \overline{n}$, the accurate limits are closer to the center line than those drawn on the chart and a point which is out of control with the drawn limits will be even more out of control with respect to the accurate limits. Similarly, when we looking at points just inside the drawn limits, we need only recalculate limits if $n_i > \overline{n}$.

as shown, a number of logs (from a number of subcontractors) is sampled, and the number of defective logs is noted. (Defects may be : wrong length, damage to the surface, wrong small-end diameter, wrongly classified, too great a degree of "sweep" and knot-holes too large). Draw a $\widehat{p}$–chart and draw in the center line and the control limits. What do you conclude? (Compute any individual control limits that you think you might need.)

Quiz for Section 13.2

1. Define the three lines used on a control chart for means.
2. At least how many subgroups should be used for constructing a control chart for means?
3. Why do we need both an $\overline{x}$–chart and an R–chart?
4. What size subgroups should we use for a $\widehat{p}$–chart?

13.3 Getting On With the Job

Generally a team approach is needed for establishing control charts. In a complex process with many parts, it is better to have several charts spread throughout the process at strategic points than a single chart at the end. Having several charts makes it easier to track down special causes when their presence has been signalled. Once a chart signals that the process is "out of control", *immediate* action is necessary to find the cause while memories of surrounding circumstances are fresh – otherwise the whole idea of a control chart is negated. It is much harder to look for special causes some time after the event than immediately. As the charts tend to be very conservative, such a signal generally means a problem. Once a signal occurs, it is very tempting to take another sample to verify the signal. This should be avoided as it can lead to accepting that the process is stable when it is not.

The choice of subgroup, which requires choosing the number and frequency of measurements and the time between subgroups, is important as measurements in a subgroup should all be affected equally if a special cause is acting. Variation within a subgroup should be largely due to common causes only. How far apart in time should measurements be made in a subgroup? The wider they are apart, the less sensitive is the mean of the subgroup in picking up changes as the effects of the changes tend to get averaged out. The mean would be most sensitive when the measurements are close together, such as consecutive observations, so that all the measurements in a subgroup are observed under as similar conditions as possible. Initially the measurements could be more widely spaced to try and pick up special causes with large effects. Once these are removed the spacing could be reduced.

The system of measurement needs to be reliable and reproducible so that different people should obtain almost identical measurements for a given object. Clearly it is a good idea to carry out a preliminary study of the measurement system.

We have noted how charts are formed using historical data, or data from a set-up phase, at which point lines are added and projected into the future.[21] Periodically, the lines would need to be updated using past data collected when the process was in control: out of control points can be ignored in the calculations. In setting up (or updating) a chart, any points outside the control limits should be checked to see if they are due to some special cause. All out-of-control points for which the cause can be identified (e.g. an operator error that is unlikely to recur) are removed and then the control lines are recalculated. When points are removed and the control lines recalculated, other points may now appear to be out of control. However, the process of removing points and recalculating limits is usually only performed once.

It is helpful to have a standard form for entering the data and constructing the chart. This will help cut down transmission errors. It is also useful to join up consecutive points, as we have done, to highlight any trends.

Quiz for Section 13.3

1. If a chart signals that the process is out of control for 1 point out of 15, which of the following is the correct action: (a) observe further points to see if the problem persists, (b) wait until the end of the run and take a look at the overall picture, or (c) look for a special cause immediately?
2. In choosing a suitable subgroup, what conditions would you like the measurements in a subgroup to satisfy?
3. How would you determine the time period between subgroups?
4. In using a control chart, when do you redraw the center line and control limits?
5. If both the $\bar{x}$–chart and the R–chart are out of control, which one should be dealt with first?

13.4   Control Charts for Single Measurements

Example 13.4.1   A company involved with the marketing of a range of plastic products has recorded the number of customer complaints on a monthly basis. The data are given in Table 13.4.1 (ignore the "|diff.|" column for the time being).   ∎

Sometimes, as in Example 13.4.1, data may be available only weekly, monthly, or even yearly so that we do not get sufficient data quickly enough to be able to form subgroups. In this case we are forced to construct a control chart using just individual measurements of the form $\widehat{\mu}_X \pm 3\widehat{\sigma}_X$. (Note, however, that if a suitable way of grouping the data is available, it is always better to use the $\bar{x}$– and R–charts as they are more sensitive in detecting special causes.)

The center line of the chart based on single observations (the "***individuals chart***") is drawn at $\widehat{\mu} = \bar{x}$, the sample mean of all the observations. The main

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[21]We do not redraw the chart after each point is calculated, which would lead to a new center line and new limits each time.

problem in constructing an individuals chart is that of estimating variability. One measure is based upon the average moving range $\overline{m}_R$ which is found by taking all the unsigned differences between consecutive measurements (e.g. the "|diff|" column in Table 13.4.1) and then averaging.[22] It turns out that we can estimate $3\sigma_X$ by $2.66\overline{m}_R$. This leads to the control lines given to the right of Fig. 13.4.1. Usually we would want at least 20 measurements.[23] The resulting chart does not detect changes in variability directly, though very long lines between consecutive points on the chart indicate instability.

Where observations are very costly to get, or the time period between observations is long, we may end up with a lot less than 20 observations. In this case it is preferable to use 2-sigma limits in the control chart rather than 3-sigma limits for greater sensitivity (i.e. use $\overline{x} \pm 1.77\overline{m}_R$).

Table 13.4.1 :   The Number of Customer Complaints per Month
for 31 Months (June 1992 to December 1994 inc.)

Month	No.	|diff.|	Month	No.	|diff.|	Month	No.	|diff.|
1	30		12	26	9	23	34	18
2	26	4	13	21	5	24	28	6
3	18	8	14	34	13	25	23	5
4	17	1	15	41	7	26	34	11
5	40	23	16	33	8	27	29	5
6	34	6	17	40	7	28	23	6
7	18	16	18	30	10	29	35	12
8	26	8	19	11	19	30	50	15
9	23	3	20	18	7	31	27	23
10	47	24	21	31	13			
11	35	12	22	52	21			

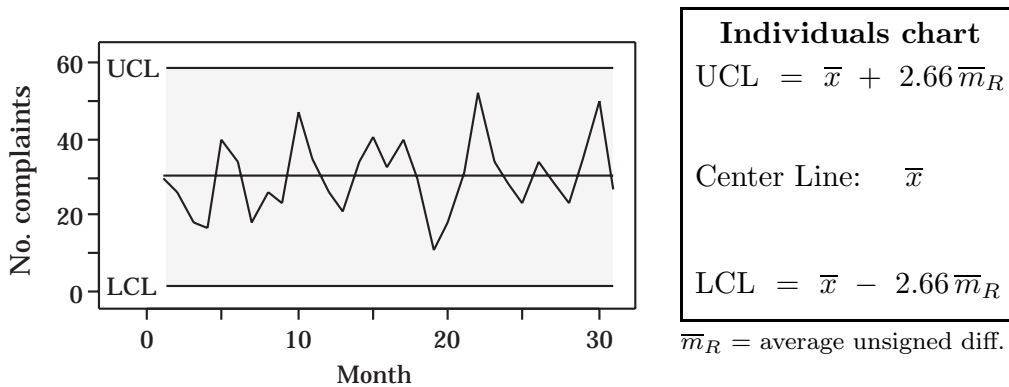|	**Individuals chart**
	UCL $= \overline{x} + 2.66\overline{m}_R$
	Center Line: $\overline{x}$
	LCL $= \overline{x} - 2.66\overline{m}_R$
	$\overline{m}_R$ = average unsigned diff.

Figure 13.4.1 :   Individuals chart for the observations in Table 13.4.1.

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[22]If there are k observations we use a divisor of $k-1$ as there are $k-1$ consecutive differences.
[23]Because of the variability of single observations, Gitlow et al. [1995] recommend at least 100 observations. However, this may not always be possible.

Example 13.4.1 cont. For the data in Table 13.4.1, the mean of the 31 observations is $\overline{x} = 30.13$ and the mean of the unsigned differences between consecutive observations (the |diff.| column in the table) is

$$\overline{m}_R = \frac{1}{31}(4 + 8 + \cdots + 23) = 10.833.$$

The control lines plotted in Fig. 13.4.1 are:

$$UCL = \overline{x} + 2.66\,\overline{m}_R = 58.95$$

and
$$LCL = \overline{x} - 2.66\,\overline{m}_R = 1.31\,.$$

From the control chart given in Fig. 13.4.1 we see that all the points lie within the control limits. This suggests that the situation is stable with regard to the number of customer complaints.

13.5 Specification Limits: **Keeping the Customer Satisfied**

The following distinctions between the purposes of specification limits and control limits are very important.

Specification limits are externally imposed by the customer.
They apply to individual units.

Control limits indicate the limits of variability, in a process under control, which we would expect from its past history.
They often apply to subgroups (e.g. subgroup mean).

A process may be in control but not satisfy the manufacturing specifications USL and LSL and vice-versa. Specification limits, which apply to individual units, should not appear on a chart constructed using subgroup means. Having all the subgroup means lying within specification limits could give a false sense of security. It is actually possible to have *all* measurements in a subgroup lying outside the specified limits but have the subgroup average within the limits. For example, the numbers -3.4, 4.5, 3.2, and -4.3 have a mean of zero.

If a process is under control but not delivering within specification limits, fundamental changes are required that are not indicated by the charts. It is often easier to change the mean level of a process than its variability. We *can* see whether USL and LSL lie outside the corresponding control limits for *single* observations, namely $\mu \pm 3\sigma$ which is estimated by $\overline{\overline{x}} \pm 3\widehat{\sigma}$. It is also helpful to compare $USL - LSL$ with $6\widehat{\sigma}$ to see if the process is *capable* of satisfying the specifications if it is centralized properly.

13.6 Looking for Departures

We have used the phrase "in control" in a technical sense only of points lying between two lines. However this does not necessarily mean that the distribution of X is actually stable. The combination of having 3-sigma limits, which are very conservative, and a small subgroup size (e.g. 3 to 6) means that the $\overline{x}$–chart will not be sensitive to small shifts in the centering of the process. In the case of an R–chart there may be a steady reduction in variability. For, example with processes where the skill of the operator has a major influence on s, the mere introduction of a control chart can often cause a reduction[24] in σ.

We will now discuss some additional indicators of out of control behavior, namely criteria other than points lying outside the control limits. If the process is stable with a distribution with mean μ that is close to being symmetric, then the probabilities of a point lying above or below the center line, respectively, are both approximately one half. Observing each point on the chart is then like tossing a fair coin, with "heads" referring to obtaining an observation above the center line and "tails" referring to below the center line. The sequence of the signs of points relative to the center line can then be represented by a sequence of heads and tails. Recall from Section 5.3 that where Y is the number of heads in a sequence of n tosses of a fair coin, then $Y \sim \text{Binomial}(n, p = 0.5)$. If there are any trends due to special causes, for example $\overline{x}$ steadily increasing so that "heads" becomes more likely, then we will start seeing runs of heads which have a low probability under the assumption of a fair coin. Grant and Leavenworth [1980] suggest looking for special causes if any of the following sequences of runs on the same side of the center line occurs: (a) 7 or more consecutive points on the same side; (b) 10 or more out of 11 consecutive points on the same side; (c) 12 or more out of 14; and (d) 14 or more out of 17 consecutive points on the same side of the center line. The probabilities of obtaining these events in a process under control are:[25] (a) 0.016; (b) 0.012; (c) 0.013; and (d) 0.013. As these probabilities are small (of the order of 1%), the occurrence of any of the above events would suggest a change in the process, possibly a shift in the process mean.

A whole range of rules for detecting other kinds of shifts have been developed and eight of these are described by Nelson [1984]. Assuming Normality, each of the eight patterns has a probability of less than 0.005 of occurring when the process is in control. His first four basic rules are:

(i) one point outside the control limits;

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[24]Often, this may just be a Hawthorne effect – a temporary change in behavior due to the awareness of being watched which tends to disappear as the subject becomes accustomed to this.

[25]All probabilities from $Y \sim \text{Binomial}(n, p = 0.5)$:
(a) $\text{pr}(Y = 0) + \text{pr}(Y = 7)$ with $n = 7$;   (b) $\text{pr}(Y \leq 1) + \text{pr}(Y \geq 10)$ with $n = 11$;
(c) $\text{pr}(Y \leq 2) + \text{pr}(Y \geq 12)$ with $n = 14$;   (d) $\text{pr}(Y \leq 3) + \text{pr}(Y \geq 14)$ with $n = 17$.
These probabilities apply to runs in the *next* sequence of points and not runs occurring *somewhere* in a longer sequence of observations. The latter are much harder to compute.

(ii) 9 points in a row above or below the center line;

(iii) 6 points in a row steadily increasing or decreasing; and

(iv) 14 points in a row alternating up and down (not necessarily crossing the center line).

The trend indicated by (ii) may suggest tool wear, improved training for the operator, or a gradual shift to new materials etc. while (iv) could indicate too much adjustment going on after each sample. The overall probability of getting a false signal from one or more of the four tests is about 1%. These rules are demonstrated in Figs 13.6.1(a)-(c). The point at which one of the above patterns has occurred is indicated by an arrow. Clearly one should look out for *any* abnormal pattern of points.
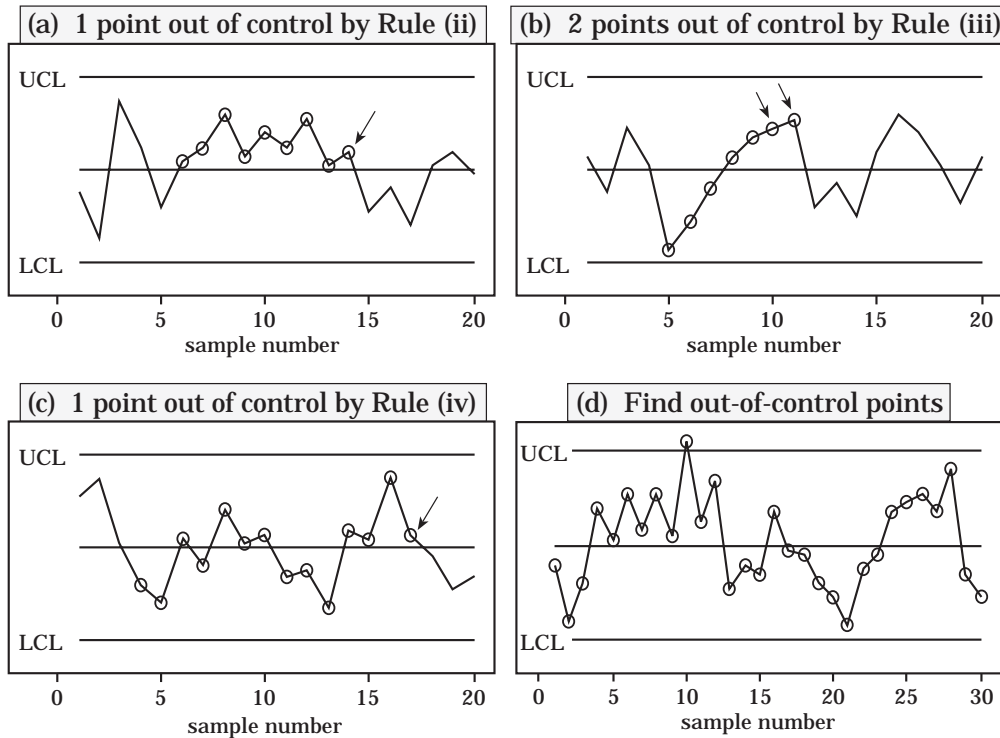
Figure 13.6.1 :   One point out of control by virtue of rule (ii).

Two further tests are suggested when it is economically desirable to have early warning. These will raise the probability of a false signal to about 2%. They are based on dividing the region between the control limits into 6 zones, each 1-sigma wide, with three above the center line and three below. The top three are respectively labeled A (outer third), B (middle third) and C (inner third), and the bottom three are the mirror image. The two additional rules are then:

(v) 2 out of 3 points in a row in zone A or beyond, and

(vi) 4 out of 5 points in a row in zone B or beyond.

Rules (v) and (vi) are appropriate when there may be more than one data source. We shall not go into details.

Clearly a variety of rules are possible and different handbooks will have some variants of the above. Also there are some redundancies in the above rules as one rule can sometimes include another. The important thing is that these formal rules should not be followed slavishly. Anything unusual in the pattern of points is a signal to go hunting for a special cause. For the exercises in this book we shall confine ourselves to just rules (i)–(iv).

Exercise on Section 13.6.1 According to the rules (i)–(iv) there are 6 points out of control in Fig. 13.6.1(d). Can you find them?

Quiz for Sections 13.4 to 13.6

  1. Describe a situation where single observations rather than averages have to be used. (Section 13.5)
  2. What is the essential distinction between specification limits and control limits?
  3. Why is it best not to include specification limits on a chart for averages? (Section 13.6)
  4. List six rules for determining whether a process is in control or not. Which four are we going to use? (Section 13.7)

13.7 Summary

  1. A **_run chart_** for a process is a graph in which the data are plotted in the order in which they were obtained in time, and consecutive points are joined by lines.

  2. Control charts are useful for
  - conveying a historical record of the behavior of a process;
  - allowing us to monitor a process for stability;
  - helping us detect changes from a previously stable pattern of variation;
  - signaling the need for the adjustment of a process;
  - helping us detect *special causes* of variation.

  3. A process is said to be **_in control_** (with respect to some measured characteristic X of the product) if the distribution of X does not change over time.

  4. There are two types of causes that produce the variation seen in a run chart:
  - **_common_** (chance) causes. These are generally small and are due to the many random elements that make up a process.
  - **_special_** (assignable) causes. These are more systematic changes in pattern for which the real cause is often found when their presence is signalled by a control chart.

5. Control charts for **subgroups** of data.
   - **x̄–chart**: plots successive subgroup means to monitor the behavior of the average level of X (see Fig. 13.2.1).
   - **R–chart**: plots successive subgroup ranges to monitor the behavior of the variability of X (see Fig. 13.2.3).
   - **p̂–chart**: plots successive subgroup proportions to monitor the behavior of the proportion of items produced which are "defective" (see Fig. 13.2.4).

6. **Individuals chart**. These are used when data from the process is produced too slowly for grouping to be feasible (see Fig. 13.4.1).

7. The upper and lower specification limits (**USL** and **LSL**) are externally imposed, for example by the customer, and are not a reflection of the behavior of the process. Also they apply to individual units, not averages.

8. The four tests we shall use for determining when a process is out of control are as follows:
   - (i) One or more points outside the control limits.
   - (ii) 9 points in a row above or below the center line.
   - (iii) 6 points in a row steadily increasing or decreasing.
   - (iv) 14 points in a row alternating up and down (not necessarily crossing the center line).

Review Exercises 13

The data sets to follow are real but their sources have not been given for reasons of confidentiality.

1. In a canning plant, dry powder is packed into cans with a nominal weight of 2000 gm. Cans are filled at a 4-head filler fed by a hopper, with each head filling about 25 cans every two minutes. The process is controlled by a computer program which calculates very crude adjustments to fill-times based on the filled weight recorded at a check weigher. Adjustments, once made, are maintained for at least 5 cans, so that a subgroup of 5 cans is a natural subgroup for studying the process. The automatic adjustment was justified by the expectation that powder density will often change because of the settling effects in the bins where it is stored prior to canning. To see how the process was performing, weights were recorded for 5 successive cans every two minutes for 60 minutes. These weights, expressed as deviations from 1984 gm, are given in Table 1.
   - (a) Why is it legitimate to work with deviations rather than the original data?[26]

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[26]Originally the observations were expressed as deviations from the target weight of 2014 gm,

(**b**) Calculate the center line and the control limits for an $\bar{x}$–chart.

(**c**) Construct the $\bar{x}$–chart. Are there any points outside the control limits?

(**d**) Construct an R–chart. What does the chart tell you?

(**e**) It turns out that because of the crude adjustment scheme, the between-subgroup variation is much greater than the within-subgroup variation. Therefore using the within subgroup variation to determine the control limits could be regarded as unreasonable. To get round this problem, treat the 30 subgroup means in the table as 30 single observations and construct an individuals control chart. How does this affect your decision about the stability of the process?

Table 1 : Canning PLant Data

Subgroup	Weight-1984 gm					Mean	Range
1	32.3	31.6	13.3	14.3	16.6	21.62	19.0
2	23.2	32.9	30.1	34.8	29.9	30.18	11.6
3	8.1	17.5	11.9	11.4	12.5	12.28	9.4
4	19.6	26.2	27.8	27.4	17.1	23.42	10.7
5	31.4	35.7	29.2	29.7	26.9	30.58	8.8
6	37.5	22.6	8.1	12.9	14.5	19.12	29.4
7	20.0	18.0	23.6	9.0	16.1	17.34	14.6
8	7.9	4.4	4.4	3.9	3.7	4.86	4.2
9	17.8	17.1	18.4	24.9	21.5	21.94	7.8
10	25.4	26.9	27.3	21.6	29.2	27.16	7.6
11	35.9	42.8	41.1	37.4	24.8	36.40	18.0
12	26.6	33.4	27.9	25.1	29.9	28.58	8.3
13	13.7	11.8	20.6	6.2	14.2	13.10	14.4
14	32.3	23.1	17.7	22.1	12.1	21.46	20.2
15	27.4	26.0	29.4	29.5	32.5	28.96	6.5
16	36.5	42.4	30.7	27.0	23.3	31.98	19.1
17	24.0	36.8	31.5	22.5	25.6	28.08	14.3
18	26.2	18.0	14.4	6.8	11.3	15.14	9.4
19	25.7	26.3	23.2	17.8	18.1	22.22	8.5
20	16.4	44.1	33.4	29.7	32.2	33.16	27.7
21	13.2	23.3	23.7	21.0	16.7	19.58	10.5
22	24.5	32.8	24.4	29.2	22.0	26.58	10.8
23	16.7	24.9	27.8	29.3	31.4	26.02	14.7
24	34.2	25.6	11.5	8.5	2.6	16.48	31.6
25	33.6	17.4	17.5	18.4	15.6	20.50	18.0
26	27.2	37.2	27.4	28.2	21.2	28.28	16.0
27	29.6	39.0	35.7	32.5	29.3	33.22	9.7
28	18.9	54.3	40.4	35.3	28.3	35.44	35.4
29	19.1	28.6	23.8	29.9	27.1	25.70	10.8
30	29.9	29.4	30.8	30.3	38.5	31.78	9.1

2. In Example 13.2.2 we saw that the $\hat{p}$–chart for the number of units reworked indicated that the production process was "out of control". A quality improvement team worked on the project and the subsequent data for the first half of 1995 are as given in Table 2.

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but this led to too may negative numbers so we added 30 to the deviations to make them all positive.

Table 2 : Units Requiring Rework After Changes

Week ending Day/month	Requiring rework	Total production	%
11/2	27	2405	1.12
18	27	2672	1.01
25	28	3057	0.92
4/3	20	3053	0.66
11	16	3062	0.52
18	5	2489	0.20
25	34	3685	0.92
1/4	25	3305	0.76
8	15	3339	0.45
15	14	2536	0.55
22	17	2701	0.63
29	30	2843	1.06
6/5	32	3809	0.84
13	15	3050	0.49
20	17	3173	0.54
27	14	3205	0.44
3/6	10	2937	0.34
10	9	2559	0.35

(a) Construct a $\widehat{p}$–chart using the average value of n.

(b) Do you think that the team was successful in stabilizing the process?

(c) What else was achieved by the team in terms of variability?

3. The data in Table 3 were obtained for a process deemed to be in control. The process produced a low-volume product, but certain numbers of the items produced resulted in credit notes being written (because of problems with the product). For this process, these were very expensive so that the sequence is short.

Table 3 : Credit Note Data

Month	Credit notes	Total no.	Month	Credit notes	Total no.
Jan	10	144	Jul	4	64
Feb	4	81	Aug	3	100
Mar	4	100	Sep	6	81
Apr	16	400	Oct	5	169
May	9	225	Nov	24	400
Jun	8	196	Dec	10	100

(a) Determine the values for the proportions defective and calculate the best value to use for the center line.

(b) Construct a run chart for the proportions defective and draw in the center line.

(c) Using the value obtained in (a) for the center line, determine *individual* 2-sigma upper and lower control limits for each point (i.e. using n_i, not $\overline{n}$.

(d) Why have we chosen "2-sigma limits" rather than the more customary 3-sigma limits?

(**e**) What do you conclude from the chart?

4. To monitor the performance of a vehicle, the mileage was recorded each time fuel was added. After 30 measurements we have the following distances (Mge.) traveled per unit volume of fuel used. The unsigned differences for consecutive pairs (|diff.|) are also given in Table 4.

Table 4 : Fuel Consumption Data

No.	Mge.	|diff.|	No.	Mge.	|diff.|	No.	Mge.	|diff.|
1	11.4		11	11.7	0.9	21	12.5	0.0
2	11.3	0.1	12	11.6	0.1	22	11.3	1.2
3	11.9	0.6	13	11.6	0.0	23	11.4	0.1
4	11.4	0.5	14	13.2	1.6	24	13.1	1.7
5	12.5	1.1	15	12.3	0.9	25	12.0	1.1
6	12.0	0.5	16	12.3	0.0	26	12.9	0.9
7	9.6	2.4	17	11.9	0.4	27	13.5	0.6
8	11.9	2.3	18	11.3	0.6	28	12.6	0.9
9	12.6	0.7	19	11.9	0.6	29	13.1	0.5
10	12.6	0.0	20	12.5	0.6	30	13.1	0.0

(**a**) What sort of control chart would you use here?

(**b**) Construct your chart and draw in the center line and the control limits. What do you conclude?

(**c**) It was found that the vehicle was used for a lot of very short trips in the city during the time when the 7th sample value was calculated. Exclude this point and recompute your limits. What do you conclude?

Table 5 : Plastic Film Data

Roll	Thick1	Thick2	Mean	Roll	Thick1	Thick2	Mean
1	115	123	119.0	26	116	126	121.0
2	114	124	119.0	27	113	123	118.0
3	115	122	118.5	28	115	124	119.5
4	114	123	118.5	29	112	122	117.0
5	114	125	119.5	30	111	124	117.5
6	115	124	119.5	31	113	123	118.0
7	112	127	119.5	32	113	120	116.5
8	115	126	120.5	33	114	124	119.0
9	113	124	118.5	34	111	125	118.0
10	113	125	119.0	35	115	124	119.5
11	117	126	121.5	36	113	124	118.5
12	115	128	121.5	37	112	124	118.0
13	116	127	121.5	38	114	127	120.5
14	113	121	117.0	39	117	127	122.0
15	112	125	118.5	40	116	126	121.0
16	113	124	118.5	41	117	127	122.0
17	114	125	119.5	42	117	127	122.0
18	113	123	118.0	43	113	125	119.0
19	116	125	120.5	44	113	125	119.0
20	114	127	120.5	45	114	126	120.0
21	112	120	116.0	46	114	129	121.5
22	110	119	114.5	47	115	129	122.0
23	117	125	121.0	48	116	129	122.5
24	116	126	121.0	49	116	130	123.0
25	113	126	119.5				

5. The data in Table 5 were collected by a factory producing rolls of plastic film. The 49 pairs of measurements represent the thickness (in microns) measured at the ends of 49 successive rolls. Each roll represents about 20 minutes of manufacturing and measurements may be taken only at the end of each roll. The two measurements from each roll are measured at two fixed positions across the film, one at the edge and one in the middle.

   (a) Construct an $\overline{x}$–chart using subgroups of size 2. Comment on the stability of the process. It looks too good to be true doesn't it! What do you think is wrong?

   (b) Using the means in the above table construct an individuals chart. Explain why this chart is more reliable than the previous one.

   (c) What does the chart show?

6. A particular product is used as a texture coating. To check on the production process 21 samples were taken over a period of 3 months in 1994. A number of measurements were carried out on each subgroup. Table 6 gives measurements on one of the variables, TSC (Total Solids Content expressed as a percentage). Construct an appropriate control chart and comment on the stability of the process.

Table 6 : Texture Coating Data

Subgroup	%TSC	Subgroup	%TSC	Subgroup	%TSC
1	49.5	8	49.1	15	49.1
2	50.9	9	49.4	16	50.0
3	49.2	10	49.6	17	50.7
4	49.1	11	49.4	18	49.7
5	49.3	12	49.5	19	50.5
6	49.5	13	51.2	20	50.3
7	50.0	14	49.6	21	50.4

7. A manufacturing company produces an emulsion. Subgroups are tested every month and the number of subgroups which do not conform to manufacturing criteria are given in Table 7 (labeled defect.).

   (a) Construct a $\widehat{p}$–chart.

   (b) Is the process stable?

8. In problem 6. above, another variable pH was measured at the same time as TSC. The data are given in Table 8. Construct a control chart and comment on the process.

Table 7 :   Emulsion Conformance Data

Month	No. defect.	Total subgroups	% defect.	Month	No. defect.	Total subgroups	%. defect.
Jan/92	13	109	11.9	Sep/93	19	123	15.5
Feb	11	108	10.2	Oct	31	126	24.6
Mar	18	135	13.3	Nov	38	142	26.8
Apr	20	112	17.9	Dec	15	90	16.7
May	14	99	14.0	Jan/94	23	106	21.7
Jun	15	97	15.5	Feb	23	115	20.0
Jul	36	108	33.3	Mar	47	136	34.6
Aug	20	100	20.0	Apr	36	98	36.7
Sep	25	121	20.7	May	39	123	31.7
Oct	28	119	23.5	Jun	28	84	33.3
Nov	34	124	27.4	Jul	44	107	41.1
Dec	21	113	18.6	Aug	29	135	21.5
Jan/93	16	104	15.4	Sep	26	108	24.1
Feb	30	102	29.4	Oct	24	112	21.4
Mar	27	131	20.6	Nov	30	113	26.5
Apr	27	122	22.1	Dec	21	102	20.6
May	29	116	25.0	Jan/95	22	93	23.6
Jun	17	96	17.7	Feb	18	132	13.6
Jul	37	118	31.4	Mar	38	130	29.2
Aug	32	122	26.3	Apr	29	113	25.7

Table 8 :   pH Data

Subgroup	pH	Subgroup	pH	Subgroup	pH
1	7.9	8	7.5	15	7.8
2	8.0	9	7.7	16	7.7
3	7.7	10	7.8	17	7.3
4	7.7	11	7.9	18	7.7
5	7.8	12	7.7	19	7.6
6	8.0	13	7.2	20	7.9
7	8.4	14	7.7	21	7.6

*9. Suppose that we are using an $\overline{x}$-chart with subgroups of size $n = 5$ in an idealized setting in which the data is Normally distributed with known mean μ and standard deviation σ. We know that $\sigma_{\overline{X}} = \text{sd}(\overline{X}) = \frac{\sigma}{\sqrt{n}}$. Use also the facts that

$$\text{pr}(\mu - 2\sigma_{\overline{X}} \le \overline{X} \le \mu + 2\sigma_{\overline{X}}) = 0.954$$

and $\qquad \text{pr}(\mu - 3\sigma_{\overline{X}} \le \overline{X} \le \mu + 3\sigma_{\overline{X}}) = 0.997.$

Assume that the process is under control and that subgroups are independent. Suppose that 100 subgroup means are plotted onto the chart over the course of a day.

(a) What is the probability that at least one subgroup mean lies outside 2-sigma control limits? Repeat for 3-sigma control limits.

(**b**) What is (**i**) the distribution, and (**ii**) the expected value of the number of subgroup means that will lie outside 2-sigma control limits? Repeat for 3-sigma control limits.

Suppose now that the process goes out of control in that the mean shifts from μ to $\mu + 0.5\sigma$. (The variability remains unchanged.)

**(c) Show that the probability that the next subgroup mean lies outside the 2-sigma control limits is 0.19 [This is the probability that the control chart signals the change with the first post-change subgroup.]

(**d**) What is the probability that at least one of the next 5 subgroup means lies outside the 2-sigma control limits?

(**e**) Repeat (c) and (d) but for 3-sigma limits.

(**f**) What have you learned from the calculations in this question?