TRAINING TEACHERS TO TEACH PROBABILITY

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In this paper we analyse the reasons why teaching probability is difficult for mathematics teachers, we describe the contents needed in the didactical preparation of teachers to teach probability and we present examples of activities to carry out this training.

Nowadays probability and statistics is part of the mathematics curricula for primary and secondary school in many countries. The reasons to include this teaching have been repeatedly highlighted over the past 20 years (e.g. Holmes, 1980; Hawkins et al., 1991; Vere-Jones, 1995), and include the usefulness of statistics and probability for daily life, its instrumental role in other disciplines, the need for a basic stochastic knowledge in many professions and its role in developing a critical reasoning. However, teaching probability and statistics is not easy for mathematics teachers. Primary and secondary level mathematics teachers frequently lack specific preparation in statistics education. For example, in Spain, prospective secondary teachers with a major in Mathematics do not receive specific training in statistics education. The situation is even worse for primary teachers, most of whom have not had basic training in statistics and this could be extended to many countries. There can be little support from textbooks and curriculum documents prepared for primary and secondary teachers, because these can frequently be misleading (Truran, & Truran 1994).

Consequently, a better prior training for teachers as well as the permanent support to these teachers from University departments and research groups is an urgent necessity. In this paper we discuss what type of didactical knowledge these teachers need, beyond the knowledge of statistics and probability itself and suggest activities that we found useful in training primary and secondary teachers at the University of Granada. We will concentrate on probability, although the main ideas are also useful in the case of statistics. As a previous step, we describe the main characteristics of stochastic knowledge and reasoning.

THE NATURE OF STOCHASTIC REASONING AND KNOWLEDGE

A main point in preparing teachers on a specific content is the epistemological reflection, which can help teachers to understand the role of the concepts within statistics and other areas, its importance in students' learning and students' conceptual difficulties in problem solving. Probability is a young area and its formal development was linked to a large number of paradoxes, which show the disparity between intuition and conceptual development in this field (Borovcnik et al., 1991). Despite the fact that Kolmogorov was able to provide a satisfactory axiomatic system in 1933, there is still some controversy about the interpretation of basic concepts, and even there still different views about the nature of probability (Fine, 1971).

Borovcnik and Peard (1996) remark that counterintuitive results in probability are found even at very elementary levels, whereas in other branches of mathematics counterintuitive results are encountered only when working at a high degree of abstraction. For example, the fact that having obtained a run of four consecutive heads when tossing a coin does not affect the probability that the following coin will result in heads is counterintuitive. Borovcnik and Peard (1996) also suggest that probabilistic reasoning is different from logical or causal reasoning. In a logical reasoning a proposition is always true or false. A proposition about a random event is true or false only after the experiment is carried out. Before carrying out the experiment we can only consider the probability of the event, and to estimate this probability we need to specify a model for the sample space of the experiment; however, we sometimes made errors in building an adequate sample space for a given experiment.

In arithmetics or geometry an operation can be reversed; for example we can add two numbers and get a result. If substracting one of the numbers we always find the same remainder no matter how many times we repeat the operation. When carrying out a random experiment we obtain different results and the results can not be reversed. Though simulation has a very important function in stabilising intuition and in materialising probabilistic problems, it does not provide the key to how and why the problems are solved. To understand the solution of a problem or to justify a result we need formal calculus, including a combinatorial scheme, which indicates the complementary nature of classical and frequentist approaches to probability.

Hawkins (1990) suggests that training in stochastics cannot be reduced to teaching conceptual structures and problem solving tools; we must also develop ways of reasoning, and a strong system of correct intuitions in the students. Stochastics is difficult to teach, because we should not only present different models and show their applications. We have to go deeper into wider questions, consisting of how to obtain knowledge from data, why a model is suitable, and deal with controversial ideas, such as randomness or causality.

Statistics is quickly moving away from pure mathematics, and being related to applications. Thus, independence is mathematically reduced to the formula of multiplying probabilities, but that definition does not include all the causality problems that subjects often relate to independence and does not always serve to decide if there is independence in a particular experiment.

WHAT IS DIDACTICAL KNOWLEDGE AND HOW CAN IT BE TAUGHT?

The teaching of statistics and probability takes place in the mathematics classroom, and teachers tend to adapt their vision of stochastics and its teaching, to problem-solving methods and reasoning standards used in mathematics. A wide statistical knowledge, even when essential is not enough for teachers to be able to teach probability. Research centred on teacher's training is producing a great deal of information about 'didactical knowledge', which includes the following complementary aspects (NCTM, 1991; Aichele and Coxford, 1994):

- Teachers' knowledge and reflection on the meaning of the concepts to be taught, and about possible ways in which this concept can be adapted to different teaching levels (e.g., reflection on the different meanings of probability and about how we can make these meanings understandable for students at different teaching levels).
- Prediction of students' learning difficulties, errors, obstacles and strategies in problem solving (e.g., prediction of students strategies in comparing two probabilities; prediction of students' confusion between the two terms in a conditional probability).
- Examples of teaching situations, teaching methodology, didactic tools and materials (e.g., challenging and interesting problems; Galton board, simulation, calculators, etc.).

It is important to find suitable and effective ways to teach this "didactical knowledge" to mathematics teachers. Today it is argued that students build their knowledge in an active way, by solving problems and interacting with their classmates. If we want teachers to follow a constructivist and social approach in their teaching, we should also use this same approach throughout their training (Even and Lappan, 1994). An important view is that we should give teachers more responsibility in their own training and help them to develop creative and critical thinking (Shulman, 1986). That is why we should create suitable conditions for teachers to reflect on their previous beliefs about teaching and discuss these ideas with other colleagues (Thompson, 1992).

Below we describe two examples of didactical activities to train teachers in probability. These activities are complementary from various viewpoints, can be used to provoke the teachers' reflection about the meaning of elementary stochastic notions, about the students' difficulties and obstacles, and about didactical methodology and materials.

SITUATION A: STUDENTS' PERCEPTION OF RANDOMNESS

In this situation we use the answers given by secondary school students to a classical item in research on the subjective perception of randomness (for a review of these investigations, see Falk and Konold, 1995). The aim is to reflect on the complexity of the meaning of stochastic notions, particularly that of randomness, show their practical utility and predict some learning difficulties.

How the idea of randomness is useful to different people

To start the activity we give the teachers the following taken from Green (1991):

Item 1: Some children were each told to toss a coin 40 times. Some did it properly. Others just

made it up. They put H for Heads and T for tails. These are Daniel and Clara's results:
Daniel: HTHTTHHTHTHHTTHTTHHTTHTHHTTHTHTHTHTHTTHT
Diana: HTTTHTTHTHTTTHTTTHHTTTHTTHTTHTTTTHTTTHT
Did Daniel or Diana make it up? How can you tell?

We explain to the teachers that the item has been extensively used in educational research to assess the secondary school students' conceptions about random results. A first question to discuss with the teachers is the following:

1) What type of people do you think are interested in problems similar to item 1?

The aim of this question is make teachers reflect on the diversity of people and institutions interested in the problem of randomness, with various purposes: Educational institutions recommend a frequentist approach to probability, suggest to experiment with "random devices", and use "random number tables". In games of chance (lotteries, etc.) people are interested in the properties of randomness because they try to win money, or they like to be implied in exciting risk situations. It is quite difficult and painstaking to obtain long sequences of random results with mechanical devices. Therefore, statisticians use random number tables, or computer programs to produce pseudo random numbers generators, and they need to assess their "quality". Scientists and professionals also use random number tables, to solve complex probabilistic problems by simulation.

Variables in assessment

The activity continues by giving the teachers information about the responses to item 1 obtained by Serrano (1993) with secondary school students, and then discussing with them the questions below.

Table 1 shows the responses obtained by Serrano (1996) to item 1, from 277 secondary school students. The results show a different trend in the answers to Daniel and Diana sequence, as well as in the answers of the two groups of students.

1. How do you explain the changes in the percentage of answers to the two sequences in the item?
2. What other characteristic would you change in the sequences to induce a change in these percentages?
3. How would you explain the differences between the two groups of students?

In spite of the similarity of the two sequences in item 1, more students in Serrano's research considered that Diana was cheating than in the case of Daniel. The students changed their response because they expect the number of heads and tails in a sequence of random results to be similar and in Diana's sequence the frequency of heads in only 12, while in Daniel's there are 19 heads.

Table 1: Frequencies and percentages of secondary school students' answers to item 1

Error! Bookmark not defined.	14 year-old (n=147)		18 year- old (n=130)	
Error! Bookmark not defined.	Daniel	Diana	Daniel	Diana
He/she made it up	54 (36.7)	83 (56.1)	30(23.1)	63 (48.5)
He/she did it correctly	86(58.1)	53 (36.1)	82 (63.0)	48 (36.9)
I don't know	7 (4.8)	11 (7.5)	18 (13.8)	19 (14.6)

We can see that slight changes in the statement of the item produce a change in students' answers. Other *task variables* to be taken into account in assessment refer to the instrument (questions used), students (capacities, knowledge interest) and situation (tools available, etc.). In this example, the differences between the two groups of students might be explained by age, but also by the fact that 18-year old students had been taught probability during their secondary education.

Students' strategies and conceptions

The percentages of positive or negative answers to item 1 in each group of students provide us with some information. However, a deeper understanding of the underlying reasoning is achieved when we analyse the arguments given by the students to support their decision. These are new questions to propose to the teachers:

Some reasons given by the students to justify that Daniel or Diana were cheating were the following:
a) The pattern of the sequence is too regular to be random, the results almost alternate;
b) The frequencies of heads and tails are too different;
c) There are too long runs; heads and tails should alternate more frequently.

4. Which of these arguments are right ? How would you explain the wrong answers?
5. Which of these arguments do you think was mainly employed in each item? What other correct and wrong arguments would you expect in this item?
6. Are these arguments similar or different to those that a professional statistician uses in testing randomness?

In the students' arguments, the complexity of the randomness is observed, since different students focus their attention on different properties of the sequences, and the correctness or incorrectness of the argument does not depend on whether the response is positive or negative, but on which sequence it refers to. Students used very sophisticated strategies. As in the case of Daniel there is a balance between the number of heads and tails, most students considered that Daniel flipped his coin correctly. Since the number of tails is more imbalanced in the second sequence, most students argued that Diana is cheating. These students computed the frequencies of heads or tails in the given sequence and compared them with the expected frequencies in a theoretical model of equiprobability. When the observed frequencies were too different from their expectation, they rejected the hypothesis that the sequence was random.

Other students analysed the length of runs. When finding a run of four or five consecutive similar results, some of them considered the sequence was not random (30.1% in Diane sequence). An underlying mechanism here is the *negative recency* explained by representativeness heuristic (Kahneman et al., 1982), where people expect a sample of random results to be very similar to the generating process, even in a short series of trials. Other possible strategies to solve the problem are:

- analysing the order of heads and tails and deciding that the sequence is too regular to be random (35.3% in item 1, 6% in item 2). Again representativeness heuristics is shown in this response;
- suggesting we cannot decide if the child is cheating, since random results are unpredictable; here students reason according *to the outcome approach* described by Konold (1989) (30.6% in item 1 and 29.6% in item 2).

These strategies are quite different from those of professional statisticians when solving a similar problem. Though in students' responses we see some intuitive elements of hypothesis testing (theoretical probability, expected frequencies in case of a true hypothesis, observed frequencies, ...), professional statisticians employ these elements in an explicit and formal way. For example, to compare the observed and expected frequencies, they would apply a goodness of fit test, choosing a significance level, and, by using the critical values of the Chi squared distribution, they would take an objective decision, as regards rejecting the sequence's randomness or not.

The number and types of strategies of professional statisticians are more complex and complete than those of students. For example, whilst a student can think he has "proven" that a sequence is random, the statistician would just say that there is no reason for rejecting the sequence randomness. Moreover, some criteria subjectively used by students to reject randomness are wrong. For example, the argument that a sequence is not random because it contains runs which are too long is, in the case of Diana, incorrect.

The complexity of randomness

When the previous analysis is finished, we can discuss with the teachers is the epistemological nature of randomness. We continue the activity with the following questions:

> 7. How can we define randomness? Do you think it is possible to speak about "absolute" randomness? Is randomness a property of some phenomena or is it a model to analyse them?
> 8. How can we know with certainty that a dice or a coin, is producing random results?

There is not a generally easy accepted definition of randomness and mathematical definitions do not allow us to determine with certainty whether a given sequence is random or not. (In Fine, 1971 different mathematical definitions of randomness are described, e.g., using computational complexity and selection algorithms). Furthermore, 'randomness' does not have the same meaning in different institutions, or for different people. In secondary school teaching, random sequences are identified with those produced by random processes. A statistician interested in building an algorithm to generate random numbers, can separate the random process and the random sequence, as he uses a determinist algorithm to generate randomness. According to Kyburg (1974) randomness is composed of the four following terms:

- the object that is supposed to be a random member of a class;
- the set of which the object is a random member (population or collective);
- the property with respect to which the object is a random member of the given class;
- the knowledge of the person giving the judgement of randomness.

For Kyburg, what is considered to be random depends on our knowledge. For example, if I have tossed a die and I observe the result obtained, this result is no longer random for me, though it may be random for another person that does not know the result. In fact, it is preferable to consider randomness as a mathematical model that we apply to some situations, because it is useful to understand them, and not as a property of these situations. A child would think that the decimal figures of the number π are random, due to their disorganised appearance. Furthermore, psychological research shows that, as a rule, subjects assign more alternations to different results than what is theoretically expected in random sequences ('*gambler fallacy*')

SITUATION B: WINNING STRATEGY IN A PROBABILISTIC GAME

In the second activity we invite teachers to play a game and to find the best possible strategy. The game is based on Bertrand's paradox and serves to compare the frequentist and Laplace conceptions of probability, and to reflect on the concepts of random dependent experiments and conditional probability, fundamental stochastic ideas (Heitele, 1975) and the role of problem-solving in the construction of mathematical knowledge.

Description of the problem-situation and classroom dynamics

We proposed that student teachers should experiment with a didactic situation that has been designed for teaching probability at secondary level. The student teachers would play the role of students and the teacher educator would take the teacher's place. We start the activity with the following game:

> We take three counters of the same shape and size. One is blue on both sides, the second is red on both sides and the third is blue on one side and red on the other. I put the three counters into a box, and I shake it, before selecting a counter at random. I'll show you one of the sides.
> 1. Can you guess the colour of the hidden side?.
> 2. Once you bet, I'll show you the hidden side. You win a point if you are right in your prediction.

Once the teachers understand the rules, and after doing some trials, I ask them to find the strategy that produces the better chances to win over a long series of trials. Each teacher will think a strategy individually, since this is a competition. The work with the teachers is organised in the following stages:

PHASE 1: Starting the game and looking for the best strategy

We do 10 trials. In each trial I pick a counter at random and show the teachers one of the sides. The teachers write down the colour they can see and then the colour predicted for the hidden side in a recording sheet (See Figure 1). Finally I show them the hidden side and they write down the colour. When the first series of ten drawings is finished, each teacher compare his score with his colleagues and we find out who are the winners.

Figure 1. Recording the game results

Trial										
Seen side										
Predicted side										
Hidden side										

PHASE 2*: Analysis of strategies and new trials*

I give some time so that the teachers can think about the strategy they used (if any) and can describe it on a sheet of paper. They might also work in pairs to propose a common strategy. Common strategies in this game are:

A: Predicting red and blue alternatively;
B: Choosing red (blue) in all the trials
C: Predicting the colour at random;
D: Two blues, one red (or something similar)
E: Predicting the colour shown on the visible side
F: Predicting the colour contrary to that shown on the visible side.

Once the teachers have described their strategies, we play 10 more trials, in which each student can choose one of the strategies above. The intention is to give the students the opportunity of expressing their ideas and check their conjectures. The results are contrasted and, when necessary, this phase is repeated to increase the total number of experiments. Generally, when increasing the number of trials, some of the strategies are discarded, because the results contradict the teachers' initial expectations. Finally, the teachers have a clear preference for one or several favourite strategies, though some of them might keep to a wrong strategy, by thinking that the failure is due to the random nature of the experiment.

PHASE 3: Justification of best strategy

The final strategies are written on the blackboard. Different teams support each strategy and discuss their arguments in favour or against them. The aim is to prove that the selected strategy is optimum; and in this attempt both the correct reasoning and the possible misconceptions are revealed. I will propose the following discussion:

3. What type of reasoning have you followed to guess that your strategy is the best?
4. Can you prove that your strategy is the best using only the empirical results?

Even when a given strategy has won in a particular series of trials, this does not prove that such a strategy will be the best in future trials. A mathematical analysis shows that E is the best strategy. This does not mean that in a series of 10, 20, ... 50 games, strategy E will certainly win, due to the random nature of the game. However, a rational person would prefer the systematic use of strategy E.

PHASE 4: Didactic reflection

The activity carried out with the teachers until this point can be used to teach some ideas of conditional probability and compound experiment in secondary level. We have then shown the teachers an example of a didactic situation, and a teaching pattern in the field of probability. At the same time this activity can reinforce the teachers' probabilistic knowledge.

Reflection on the mathematical content

In the problem posed -looking for the best strategy in a game- we can work with different types of knowledge. It is important that teachers identify what the type of reasoning that serves to validate the best strategy is and compare empirical confirmation as a support for decision with logical or combinatorial deductive arguments which can "prove" our solution.

It is possible to apply here the equiprobability principle, the classical Laplace's conception of probability of an event as the 'quotient between the number of cases favourable to the event and the number of possible cases' is useful. However, the subject has to conventionally assume the symmetry of the elementary events, and the independence of successive experiments, which often requires a subjective judgement. If, instead of using the counter, we had used a biased die or a thumbtack in this game, the frequentist conception of probability would be preferable. In this conception, probability is the limit to which the relative frequency of the event would tend over a long series of trials. However, it is necessary to be aware of the possibility of oscillations in the relative frequency, especially, for a limited number of trials.

Furthermore, once the probabilities of the elementary events were known, we could apply probabilistic and combinatorial rules again to calculate more complex probabilities. However, in the situations where probabilities calculus is too complex, simulation allows us to obtain an estimate for the probabilities of events, especially when the number of trials is high enough. Consequently, this game contextualizes the debate between different conceptions of probability, and shows the insufficiency and complementary nature of each of these conceptions.

In this situation - and, moreover, in any simple random situation, the fundamental stochastic ideas described by Heitele (1975) appear. Fundamental ideas could be present through the curriculum from the school level until university, with various degrees of formalisation. Besides the ideas of event, probability and convergence, Heitele mentions combinatorial operations, the addition and multiplication of probabilities, independence conditional probability, random variable, equidistribution and symmetry, expectation and sampling. All of these ideas appear in this game and the analysis of the game will serve to reflect with teachers about which are the fundamental stochastic ideas, and what degree of formalisation of the same is appropriate to present these ideas to secondary students.

FINAL REMARKS

The examples described are complementary from different perspectives. As regards randomness, the first situation starts out from an experiment that has already been carried out, and the randomness must be judged after the data has been obtained ('a posteriori' statistical study of the experiment). In the second situation, the subject is asked for a prediction of the results of the experiment, from analysing the structure of the problem-situation ('a priori' probabilistic study of the experiment). The two situations analysed also show examples of different visions of mathematics:

- The *formal* vision *of mathematical knowledge,* which serves to validate the best strategy in the game using an existing mathematical theory, in this case, combinatorics;
- The *empirical* vision*,* which emphasises the role of experimenting in mathematics, and the type of validation that it provides: a mathematical solution (a strategy) is validated through statistical knowledge, when its use provides better results in the long run.
- The *structural - analytical* vision: the trials sequence is judged to be random through the analysis of its structure and comparison with the properties of the mathematical model expected.

The student is less involved in the first activity (to decide if another child was cheating) than the second one (to play and choose a strategy for winning a game). This second activity also shows the difficulty of matching students' attitudes and intuitions with their formal thinking, in the field of probability.

An important conclusion is that the didactic training of teachers must show them how to carry out didactic analyses similar to those presented here. The two situations described for the case of probability may serve as examples for designing similar situations referring to other mathematical contents. This type of analysis could be the core for teacher training courses, from the mathematical and didactic point of view.

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