Prove by induction: $\sum_{k=1}^{n} 2^k = 2(2^n - 1)$ $(*)$ $(n=1,2,3,\ldots)$

Base case: $n=1$. R.T.P. $\sum_{k=1}^{1} 2^k = 2(2^1 - 1)$ (1)

$$\begin{aligned}
\text{LHS of (1)} &= \sum_{k=1}^{1} 2^k \\
&= 2^1 \\
\text{RHS of (1)} &= 2(2^1 - 1) \\
&= 2(1) \\
&= 2 \\
&= \text{LHS of (1)}.
\end{aligned}$$

So base case $n=1$ is proved.

Prove by induction: $\sum_{k=1}^{n} 2^k = 2(2^n - 1)$ ⊛ $(n = 1, 2, 3, \ldots)$

General case: assume ⊛ is true for some $n = x$:

so we can assume $\sum_{k=1}^{x} 2^k = 2(2^x - 1)$. ⓐ

R.T.P. $\sum_{k=1}^{x+1} 2^k = 2(2^{x+1} - 1)$. (**) .

$$
\begin{aligned}
\text{LHS of } (**) &= \sum_{k=1}^{x+1} 2^k \\
&= \sum_{k=1}^{x} 2^k + 2^{x+1} \\
&= 2(2^x - 1) + 2^{x+1} \quad \text{by allowed ⓐ} \\
&= 2^{x+1} - 2 + 2^{x+1} \\
&= 2 * 2^{x+1} - 2 \\
&= 2(2^{x+1} - 1) \\
&= \text{RHS of } (**).
\end{aligned}
$$

So if ⊛ is true for $n = x$, we have proved ⊛ true for $n = x + 1$.

⊛ is true for base case $n = 1$,

so we have proved ⊛ true for all $n = 1, 2, 3, \ldots$. □