Let $y = \dfrac{1}{a-bx}$. Prove $\dfrac{d^n y}{dx^n} = \dfrac{n!\, b^n}{(a-bx)^{n+1}}$ $(*)$ for $n=1, 2, 3, \ldots$

Base case: $n=1$

$$\text{LHS} = \frac{dy}{dx} = \frac{d}{dx}\left\{(a-bx)^{-1}\right\} = -(a-bx)^{-2}(-b)$$
$$= \frac{b}{(a-bx)^2}.$$

$$\text{RHS} = \frac{1!\, b}{(a-bx)^{1+1}} = \frac{b}{(a-bx)^2} = \text{LHS}.$$

So $(*)$ is proved for base case $n=1$.

Let $y = \dfrac{1}{a-bx}$. Prove $\dfrac{d^n y}{dx^n} = \dfrac{n!\, b^n}{(a-bx)^{n+1}}$ (*) for $n = 1, 2, 3, \dots$

General case: suppose (*) is true for $n=k$ for some k.

So we may assume $\dfrac{d^k y}{dx^k} = \dfrac{k!\, b^k}{(a-bx)^{k+1}}$. (a)

R.T.P. (*) true for $n = k+1$:

RTP $\dfrac{d^{k+1} y}{dx^{k+1}} = \dfrac{(k+1)!\, b^{k+1}}{(a-bx)^{k+2}}$. (**)

$$\text{LHS of } (**) = \frac{d^{k+1}y}{dx^{k+1}}$$

$$= \frac{d}{dx}\left\{\frac{d^k y}{dx^k}\right\}$$

$\dfrac{1}{(a-bx)^{k+1}} = (a-bx)^{-(k+1)}$

$$= \frac{d}{dx}\left\{k!\, b^k (a-bx)^{-k-1}\right\} \quad \text{by allowed (a)}$$

$$= k!\, b^k (-k-1)(a-bx)^{-(k+2)}(-b)$$

$$= \frac{k!\, b^k (k+1) b}{(a-bx)^{k+2}}$$

$$= \frac{(k+1)!\, b^{k+1}}{(a-bx)^{k+2}}$$

$$= \text{RHS of } (**).$$

So if (*) is true when $n=k$, then (*) is proved true for $n = k+1$ also.

(*) is true for base case $n=1$, so we have proved (*) true for all $n = 1, 2, 3, \dots$. $\square$