On Bayesian Semiparametric Quantile Regression

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Outline of this document

1. Quantile Regression
   - Quantile Regression and Asymmetric Laplace distribution

2. Semiparametric Regression and Graphical Model
   - Penalised Spline
   - Graphical Models

3. Bayesian Quantile Regression

4. Extension
Quantile Regression

Introduction

Some motivation

- For a regression model, typically, we use the quadratic loss or absolute loss function and it means, we look at the $E[Y|X]$ or $Median[Y|X]$, respectively.

- But sometimes we lose some information with these models and they are not appropriate for all data. In addition, in some cases the tails of the distributions are more interest than the center of them.

- To give a more complete picture of the relationship between the response $Y$ and explanatory variables $X$, we can consider the quantiles of distribution of $[Y|X]$ i.e. $Q[Y|X]$.

- The resulting curves are called the quantile regression curves. Clearly, they can be smoothed in some ways.
Figure: \((X, Y)\) have the bivariate normal distribution
\[
Q(y|x)(p) = \mu(y|x) + \sigma(y|x) \Phi^{-1}(p).
\]
Figure: $(X, Y)$ have the bivariate normal distribution

$$Q(y|x)(p) = \mu(y|x) + \sigma(y|x) \Phi^{-1}(p)$$
Quantile Regression

Growth chart example

Girls’ height and weight.
Here are three classes

1. **Classical Quantile Regression** models. Using Asymmetric $L_1$ loss function, Koenker and Bassett (1978).


3. **LMS-type** methods. These transform the response to some parametric distribution, Cole and Green (1992) (e.g., Box-Cox to $N(0, 1)$).
Classical Quantile Regression

Koenker and Bassett (1978) considered asymmetric $L_1$ loss function

$$
\rho_p(u) = \begin{cases} 
(1 - p)(-u), & u < 0, \\
p(u), & u \geq 0,
\end{cases}
$$

Figure: (a) are symmetric and asymmetric absolute loss function with $p = 0.5$ ($L_1$ regression) and $p = 0.75$ (asymmetric $L_1$ regression); (b) are the derivatives of loss functions or Influence functions of (a).
Quantile Regression (QR) and AL distribution

- Suppose the $p$th conditional quantile is $Q_{(Y|X)}(p) = X\beta^{(p)}$. 

Estimation

$$
\hat{\beta}^{(p)} = \arg \min_{\beta \in \mathbb{R}^k} \sum_{i=1}^{n} \rho_p(y_i - x_i^T \beta)
$$

Quantiles traditionally are estimated by linear programming. Koenker and Machado (1999) considered the following representation of asymmetric Laplace distribution (AL) in QR.

AL distribution

$$
f(y; \mu, \sigma, p) = p(1-p)\sigma \exp\left\{-\rho_p\frac{y-\mu}{\sigma}\right\} \quad (1)
$$

$\mu \in \mathbb{R}$, $\sigma > 0$, $0 < p < 1$ and $P(Y \leq \mu) = p$. 
Quantile Regression (QR) and AL distribution

- Suppose the $p$th conditional quantile is $Q_{(Y|X)}(p) = X\beta^{(p)}$.

**Estimation**

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**AL distribution**

$$f(y; \mu, \sigma, p) = \frac{p(1 - p)}{\sigma} \exp \left\{ -\rho_p \left( \frac{y - \mu}{\sigma} \right) \right\} \tag{1}$$

- $\mu \in \mathbb{R}$, $\sigma > 0$, $0 < p < 1$ and $P(Y \leq \mu) = p$. 

Quantile Regression and AL distribution

Consider the linear model

\[ y_i = x_i^t \beta + \epsilon_i, \quad \text{for } i = 1, \ldots, n, \]

and let

\[ \epsilon_i \sim AL(0, \sigma, p) \quad \text{for } i = 1, \ldots, n. \]

\[ L(\beta) = \prod_{i=1}^{n} f(y_i; \beta, \sigma) = \exp \left\{ \sum_{i=1}^{n} \rho \left( \frac{y_i - x_i^t \beta}{\sigma} \right) \right\}, \quad (2) \]

\[ \rho(x) = -\log(f(x)), \quad (3) \]

\[ \hat{\beta}_n^{(p)} = \arg \min \left\{ \sum_{i=1}^{n} \rho \left( \frac{y_i - x_{ni}^t \beta}{\sigma} \right) : \beta \in \mathbb{R}^q \right\}. \]
\[ Q(Y|X)(0.75) = a^{(0.75)} + b^{(0.75)}x \]
Next
First, Penalised Spline.
Then, Hierarchical Bayesian Modelling
and Graphical Models.
After that, an example.
Penalised spline

Consider

\[ y_i = f(x_i) + \varepsilon_i \quad \text{where} \quad f(x_i) = E(Y_i|x_i). \]

Penalised spline approach is:

\[ f(x_i) = \beta_0 + \beta_1 x_i + \sum_{k=1}^{K} u_k z_k(x_i) \]
Penalised spline

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where

- the \( u_k \) are penalised coeffs and
- \( z_k(x_i) \) are spline basis functions.
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One way to penalise is

\[ u_k \sim N(0, \sigma_u^2). \]

Our default here is “O’Sullivan spline” for the \( z_k \) which provides a close approximation to smoothing splines. Wand & Ormerod (2008).
Bracket notation:

Conditional distribution

\[ [y|x] = \text{density of } y \text{ given } x, \]

e.g.

\[ [x|\alpha, \beta] = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp \left( -\frac{x}{\beta} \right) \]

means

\[ f_X(x; \alpha, \beta) = [x; \alpha, \beta] \]

or

\[ [x] \sim \text{Gamma}(\alpha, \beta). \]
Hierarchical Bayesian Modelling

For ordinary nonparametric regression

$$[y_i | \beta_0, \beta_1, u_1, u_2 \ldots, u_K, \sigma_u^2, \sigma_\epsilon^2] \sim \mathcal{N}(\beta_0 + \beta_1 x + \sum_{i=1}^{K} u_k z_k(x_i), \sigma_\epsilon^2)$$
Hierarchical Bayesian Modelling

For ordinary nonparametric regression

Model

\[ y_i | \beta_0, \beta_1, u_1, u_2 \ldots, u_k, \sigma_u^2, \sigma_\epsilon^2 \sim N(\beta_0 + \beta_1 x + \sum_{i=1}^{K} u_k z_k(x_i), \sigma_\epsilon^2) \]

\[ [u_k | \sigma_u^2] \sim N(0, \sigma_u^2), \]

\[ [\beta_0] \sim N(0, 10^8), \quad [\beta_1] \sim N(0, 10^8), \]
Hierarchical Bayesian Modelling

For ordinary nonparametric regression

Model

$$[y_i | \beta_0, \beta_1, u_1, u_2 \ldots, u_k, \sigma_u^2, \sigma_\epsilon^2] \sim N(\beta_0 + \beta_1 x + \sum_{i=1}^{K} u_k z_k(x_i), \sigma_\epsilon^2)$$

- $$[u_k | \sigma_u^2] \sim N(0, \sigma_u^2),$$
- $$[\beta_0] \sim N(0, 10^8),$$
- $$[\beta_1] \sim N(0, 10^8),$$

- $$[\sigma_\epsilon^2], [\sigma_u^2] \sim \text{Inverse Gamma}(\frac{1}{100}, \frac{1}{100})$$

or

- $$[\sigma_u^2] \sim \text{Half Cauchy}(\text{scale} = 25)$$
Hierarchical Bayesian Modelling

Matrix Notation

\[ X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \quad \text{and} \quad Z = [z_k(x_i)]. \]

\[
[y \mid \beta, u, \sigma_e^2, \sigma_u^2] \sim N(X\beta + Zu, \sigma_e^2 I), \\
[u \mid \sigma_u^2] \sim N(0, \sigma_u^2 I), \\
[\beta] \sim N(0, 10^8 I).
\]
Hierarchical Bayesian Modelling

Matrix Notation

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\quad \text{and} \quad
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\[
[y|\beta, u, \sigma^2_{\epsilon}, \sigma^2_u] \sim N(X\beta + Zu, \sigma^2_{\epsilon}I),
\]

\[
[u|\sigma^2_u] \sim N(0, \sigma^2_u I),
\]

\[
[\beta] \sim N(0, 10^8 I).
\]

For estimating,

\[q_p(x) = p^{th} \text{ quantile of } y \text{ given } x,\]

we just have

\[
[y|\beta, u, \sigma^2_{\epsilon}, \sigma^2_u] \sim AL(X\beta + Zu, \sigma^*_{\epsilon}, p).
\]
Graphical Models

A graphical model is
Graphical Models

A graphical model is a probabilistic model for which a graph denotes the conditional independence structure between random variables.
Graphical Models

A graphical model is a probabilistic model for which a graph denotes the conditional independence structure between random variables. They are commonly used in probability theory, statistics—particularly Bayesian statistics—and machine learning.
Graphical Models

Graphical models (probabilistic graphical models) framework is also very useful for semiparametric regression, especially when the problem is non-standard.
Graphical Models

- **Graphical models** (probabilistic graphical models) framework is also very useful for semiparametric regression, especially when the problem is non-standard.
- There are two main types:
  1. Directed acyclic graphs (DAGs), also known as Bayesian networks,
  2. Undirected graphs, also known as Markov random fields.
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  - Non-random nodes are shown as small solid circles. ●
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We use the same conventions as Bishop (2006).

- Random nodes are denoted by open circles.
- Non-random nodes are shown as small solid circles.
- Observed (“evidence”) nodes are distinguished from parameter (“hidden”) nodes using shading.
DAG = Directed Acyclic Graph
Joint density function of $x_1, \ldots, x_5$

$$\equiv \left[ x_1, x_2, x_3, x_4, x_5 \right]$$

$$= \left[ x_1 \right] \left[ x_2 \right] \left[ x_3 \mid x_1, x_2 \right] \left[ x_4 \mid x_2 \right] \left[ x_5 \mid x_3, x_4 \right]$$
DAGs and Hierarchical Bayesian Models

Bayesian simple quantile regression:

\[
[y_i | \beta_0, \beta_1, \sigma^2] \overset{\text{ind.}}{\sim} AL(\beta_0 + \beta_1 x, \sigma, p), \quad 1 \leq i \leq n,
\]

\[
[\beta_0] \sim N(\mu_{\beta_0}, \sigma_{\beta_0}^2), \quad [\beta_1] \sim N(\mu_{\beta_1}, \sigma_{\beta_1}^2),
\]

\[
[\sigma^2] \sim \text{Inverse Gamma}(A, B) = IG(A, B).
\]
Bayesian Quantile Regression

\[ \mu_{\beta_0}, \sigma_{\beta_0} \]
\[ \mu_{\beta_1}, \sigma_{\beta_1} \]
\[ A, B \]

\[ \beta_0, \beta_1 \]

\[ y_i \]
\[ x_i \]
\[ n \]
Hierarchical Bayes Model for QR Model:

Bayesian quantile Model:

\[
[y_i|x_i, \beta, u, \sigma_\epsilon] \overset{ind.}{\sim} AL(\beta_0 + \beta_1 x_i + \sum_{k=1}^{K} u_k z_k(x_i), \sigma_\epsilon, p),
\]

\[
[U|\sigma_u^2] \sim N(0, \sigma_u^2 I)
\]

\[
[\beta_0] \sim N(\mu_{\beta_0}, \sigma_{\beta_0}^2), \quad [\beta_1] \sim N(\mu_{\beta_1}, \sigma_{\beta_1}^2),
\]

\[
[\sigma_u^2] \sim IG(A_u, B_u), \quad [\sigma_\epsilon^2] \sim IG(A_\epsilon, B_\epsilon).
\]
Example: The Canadian age–income data

Scatterplot of \( \log(\text{income}) \) versus age for a sample of \( n = 205 \) Canadian workers with posterior (solid) and 95% credible intervals for the 25% quantile regression.
<table>
<thead>
<tr>
<th>parameter</th>
<th>trace</th>
<th>lag 1</th>
<th>acf</th>
<th>density</th>
<th>summary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_\epsilon$</td>
<td><img src="image1" alt="trace" /></td>
<td><img src="image2" alt="lag1" /></td>
<td><img src="image3" alt="acf" /></td>
<td><img src="image4" alt="density" /></td>
<td>posterior mean: 0.161 95% credible interval: (0.14,0.185)</td>
</tr>
<tr>
<td>$\sigma_U$</td>
<td><img src="image5" alt="trace" /></td>
<td><img src="image6" alt="lag1" /></td>
<td><img src="image7" alt="acf" /></td>
<td><img src="image8" alt="density" /></td>
<td>posterior mean: 0.101 95% credible interval: (0.0621,0.159)</td>
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<tr>
<td>degrees of freedom for f</td>
<td><img src="image9" alt="trace" /></td>
<td><img src="image10" alt="lag1" /></td>
<td><img src="image11" alt="acf" /></td>
<td><img src="image12" alt="density" /></td>
<td>posterior mean: 17.1 95% credible interval: (14.1,20.4)</td>
</tr>
<tr>
<td>first quartile of x</td>
<td><img src="image13" alt="trace" /></td>
<td><img src="image14" alt="lag1" /></td>
<td><img src="image15" alt="acf" /></td>
<td><img src="image16" alt="density" /></td>
<td>posterior mean: 13.4 95% credible interval: (13.2,13.5)</td>
</tr>
<tr>
<td>second quartile of x</td>
<td><img src="image17" alt="trace" /></td>
<td><img src="image18" alt="lag1" /></td>
<td><img src="image19" alt="acf" /></td>
<td><img src="image20" alt="density" /></td>
<td>posterior mean: 13.5 95% credible interval: (13.3,13.7)</td>
</tr>
<tr>
<td>third quartile of x</td>
<td><img src="image21" alt="trace" /></td>
<td><img src="image22" alt="lag1" /></td>
<td><img src="image23" alt="acf" /></td>
<td><img src="image24" alt="density" /></td>
<td>posterior mean: 13.5 95% credible interval: (13.3,13.7)</td>
</tr>
</tbody>
</table>
Now,

A Semiparametric Regression model.
Spinal Bone Mineral Data

Bayesian Quantile Regression

Spinal Bone Mineral Density

ethnicity = 0
ethnicity = 1

age (years)

spinal bone mineral density

10 15 20 25
Model for Spinal Bone Mineral Data and its DAG

Bayesian semiparametric quantile model:

\[
\begin{align*}
[y_{ij} | \beta, u_{sbj}, u_{spl}, \sigma^2_{sbj}, \sigma^2_{spl}, \sigma_\epsilon] & \overset{\text{ind.}}{\sim} AL(\beta^T x_i + u_{i,sbj} + f(\text{age}_{i,j}; \sigma^2_{spl}), \sigma_\epsilon, p), \\
[u_{sbj} | \sigma^2_{sbj}] & \sim N(0, \sigma^2_{sbj} I), \\
[u_{spl} | \sigma^2_{spl}] & \sim N(0, \sigma^2_{spl} I), \\
[\beta] & \sim N(0, \sigma^2_{\beta} I), \\
[\sigma^2_{sbj}] & \sim IG(A_{sbj}, B_{sbj}), \\
[\sigma^2_{spl}] & \sim IG(A_{spl}, B_{spl}), \\
[\sigma^2_\epsilon] & \sim IG(A_\epsilon, B_\epsilon).
\end{align*}
\]
Nonparametric Quantile Regression with Missingness in Predictor

\[ y_i = f(x_i) + \varepsilon_i, \quad \varepsilon_i \text{ i.i.d. } AL(0, \sigma^2_{\varepsilon}, p), \quad 1 \leq i \leq n \]

\[ x_i \overset{ind.}{\sim} N(\mu_x, \sigma^2_x), \text{ but some are missing} \]

(completely at random).
Hierarchical Bayes Model for Missingness Example

Bayesian quantile Model:

\[ [y_i|x_i, \beta, u, \sigma_\epsilon] \overset{\text{ind.}}{\sim} AL(\beta_0 + \beta_1 x_i + \sum_{k=1}^K u_k z_k(x_i), \sigma_\epsilon, p), \]

\[ [x_i|\mu_x, \sigma_x^2] \sim N(\mu_x, \sigma_x^2), \quad [U|\sigma_u^2] \sim N(0, \sigma_u^2 I) \]

\[ [\beta] \sim N(0, \sigma_\beta^2 I), \quad [\mu_x] \sim N(0, \sigma_{\mu_x}^2), \]

\[ [\sigma_x^2] \sim IG(A_x, B_x), [\sigma_u^2] \sim IG(A_u, B_u), [\sigma_\epsilon^2] \sim IG(A_\epsilon, B_\epsilon). \]
Bayesian Quantile Regression

On Bayesian Semiparametric Quantile Regression

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Example on Missing Data

Bayesian Quantile Regression

- Fully observed
- $x$ value unobserved

true $f$

25 %

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<table>
<thead>
<tr>
<th>parameter</th>
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<th>density</th>
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<tbody>
<tr>
<td>$\mu_x$</td>
<td><img src="trace1.png" alt="Trace Plot" /></td>
<td><img src="scatter1.png" alt="Scatter Plot" /></td>
<td><img src="acf1.png" alt="ACF Plot" /></td>
<td><img src="density1.png" alt="Density Plot" /></td>
<td>posterior mean: 0.476 95% credible interval: (0.455, 0.497)</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td><img src="trace2.png" alt="Trace Plot" /></td>
<td><img src="scatter2.png" alt="Scatter Plot" /></td>
<td><img src="acf2.png" alt="ACF Plot" /></td>
<td><img src="density2.png" alt="Density Plot" /></td>
<td>posterior mean: 0.161 95% credible interval: (0.147, 0.177)</td>
</tr>
<tr>
<td>$\sigma_\varepsilon$</td>
<td><img src="trace3.png" alt="Trace Plot" /></td>
<td><img src="scatter3.png" alt="Scatter Plot" /></td>
<td><img src="acf3.png" alt="ACF Plot" /></td>
<td><img src="density3.png" alt="Density Plot" /></td>
<td>posterior mean: 0.109 95% credible interval: (0.0961, 0.124)</td>
</tr>
<tr>
<td>degrees of freedom for $f$</td>
<td><img src="trace4.png" alt="Trace Plot" /></td>
<td><img src="scatter4.png" alt="Scatter Plot" /></td>
<td><img src="acf4.png" alt="ACF Plot" /></td>
<td><img src="density4.png" alt="Density Plot" /></td>
<td>posterior mean: 19.1 95% credible interval: (17.1, 21.4)</td>
</tr>
<tr>
<td>first quartile of $x$</td>
<td><img src="trace5.png" alt="Trace Plot" /></td>
<td><img src="scatter5.png" alt="Scatter Plot" /></td>
<td><img src="acf5.png" alt="ACF Plot" /></td>
<td><img src="density5.png" alt="Density Plot" /></td>
<td>posterior mean: -1.1 95% credible interval: (-1.16, -1.03)</td>
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<tr>
<td>second quart. of $x$</td>
<td><img src="trace6.png" alt="Trace Plot" /></td>
<td><img src="scatter6.png" alt="Scatter Plot" /></td>
<td><img src="acf6.png" alt="ACF Plot" /></td>
<td><img src="density6.png" alt="Density Plot" /></td>
<td>posterior mean: -0.734 95% credible interval: (-0.804, -0.666)</td>
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<tr>
<td>third quartile of $x$</td>
<td><img src="trace7.png" alt="Trace Plot" /></td>
<td><img src="scatter7.png" alt="Scatter Plot" /></td>
<td><img src="acf7.png" alt="ACF Plot" /></td>
<td><img src="density7.png" alt="Density Plot" /></td>
<td>posterior mean: 0.556 95% credible interval: (0.459, 0.678)</td>
</tr>
<tr>
<td>parameter</td>
<td>trace</td>
<td>lag 1</td>
<td>acf</td>
<td>density</td>
<td>summary</td>
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<tr>
<td>$x_{mis}^{10}$</td>
<td><img src="image" alt="_trace" /></td>
<td><img src="image" alt="lag_1" /></td>
<td><img src="image" alt="acf" /></td>
<td><img src="image" alt="density" /></td>
<td>posterior mean: 0.543  95% credible interval: (0.133,0.696)</td>
</tr>
<tr>
<td>$x_{mis}^{18}$</td>
<td><img src="image" alt="trace" /></td>
<td><img src="image" alt="lag_1" /></td>
<td><img src="image" alt="acf" /></td>
<td><img src="image" alt="density" /></td>
<td>posterior mean: 0.427  95% credible interval: (0.297,0.895)</td>
</tr>
<tr>
<td>$x_{mis}^{27}$</td>
<td><img src="image" alt="trace" /></td>
<td><img src="image" alt="lag_1" /></td>
<td><img src="image" alt="acf" /></td>
<td><img src="image" alt="density" /></td>
<td>posterior mean: 0.507  95% credible interval: (0.178,0.756)</td>
</tr>
<tr>
<td>$x_{mis}^{44}$</td>
<td><img src="image" alt="trace" /></td>
<td><img src="image" alt="lag_1" /></td>
<td><img src="image" alt="acf" /></td>
<td><img src="image" alt="density" /></td>
<td>posterior mean: 0.489  95% credible interval: (0.223,0.773)</td>
</tr>
<tr>
<td>$x_{mis}^{59}$</td>
<td><img src="image" alt="trace" /></td>
<td><img src="image" alt="lag_1" /></td>
<td><img src="image" alt="acf" /></td>
<td><img src="image" alt="density" /></td>
<td>posterior mean: 0.467  95% credible interval: (0.252,0.783)</td>
</tr>
</tbody>
</table>
Extension

1. Longitudinal data see Wand (2003)
2. Additive models
3. Additive mixed models
4. Measurement error
Thank You