Solve: $m_{x+2} - m_{x+1} + m_x = c_x$ or similar eqn. (E)

$c_x$ could be 0 or could be any fn of $x$ (including a constant)

## Homogeneous Equation:

$$u_{x+2} - u_{x+1} + u_x = 0 \qquad (H)$$

For $x = 0, 1, 2, \ldots$

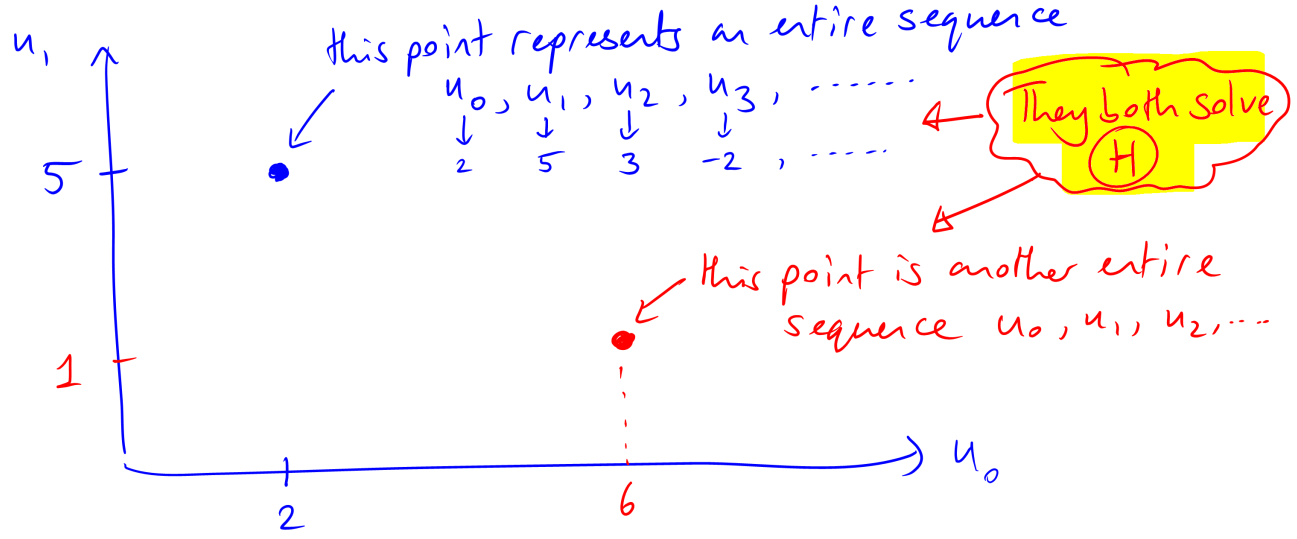
Consider a solution of this: let $u_0 = 2$, let $u_1 = 5$ } any numbers you like

Then $u_2 - u_1 + u_0 = 0 \Rightarrow u_2 - 5 + 2 = 0 \Rightarrow u_2 = 3$.

Then $u_3 - u_2 + u_1 = 0 \Rightarrow u_3 - 3 + 5 = 0 \Rightarrow u_3 = -2$

etc etc.

So if we start with any numbers $u_0 \in \mathbb{R}$, $u_1 \in \mathbb{R}$, we get a complete solution $u_0, u_1, u_2, u_3, u_4, \ldots$



Imagine I have
Two solutions: $u_{x+2} - u_{x+1} + u_x = 0$

$$v_{x+2} - v_{x+1} + v_x = 0$$

$$\Rightarrow (u_{x+2}+v_{x+2}) - (u_{x+1}+v_{x+1}) + (u_x+v_x) = 0$$

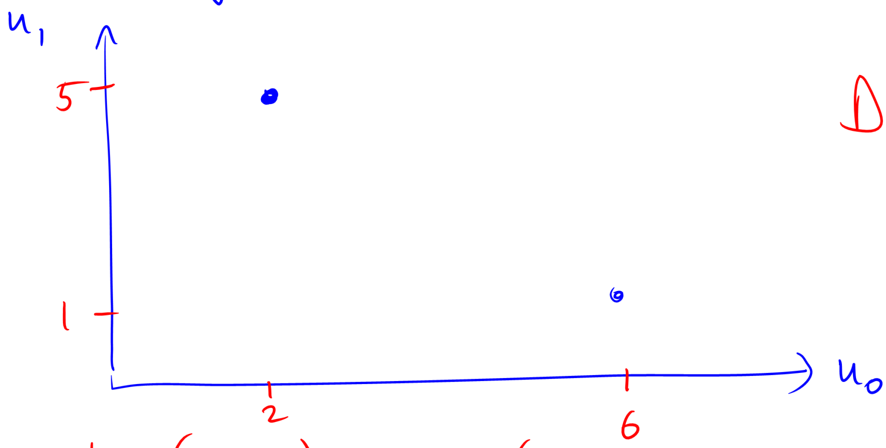
$\Rightarrow$ if $w_x = u_x + v_x$ for all $x$, then

$\{u_x\}, \{v_x\}$ both solutions $\Rightarrow \{w_x\}$ solution.

Also $u_0 = u_1 = u_2 = \dots = 0$ is a solution.

It's becoming clear that the space of solutions of (H) is a vector space.

So if we could find a basis for the vector space of solutions, then ALL possible solutions MUST be a linear combination of those basis solutions. Yippee!! ☺




Dimension of this vector space of solutions?

Dimension = 2, because every possible solution has to be a point in 2-d space, and vice versa.

The points $(2,5)$ and $(6,1)$ are a "basis" but we are greedy, we want a mathematical formula for the solution $u_x$ for any $x$.

e.g. we want to be able to say $u_x = \left(\frac{1}{2}\right)^x$ for all $x$ (like one of the 325 Ass 1 solutions), or whatever the correct answer is.

At this point, we rely on lucky guesses (fortunately, well-known ones).

Guess: solution of the form $u_x = t^x$ for some $t$ to be found.

Then to be a solution, we need

$$u_{x+2} - u_{x+1} + u_x = 0 \qquad (\text{eqn } H)$$

So if $u_x = t^x$ for all $x$, then:

$$t^{x+2} - t^{x+1} + t^x = 0 \quad \text{for all } x$$

$$t^x \{ \underbrace{t^2 - t + 1}_{\text{characteristic equation}} \} = 0$$

So either $t^x = 0$ for all $x$ $\longrightarrow$ not useful for a basis solution

or $t^2 - t + 1 = 0$ $\longrightarrow$ so this is what we want!

Moreover, it gives us not just ONE solution, but TWO!! (as long as the quadratic eqn has two distinct roots).

So if we solve this eqn, we get:

$$t_1 = \frac{1 + \sqrt{1-4}}{2} \qquad t_2 = \frac{1 - \sqrt{1-4}}{2}$$

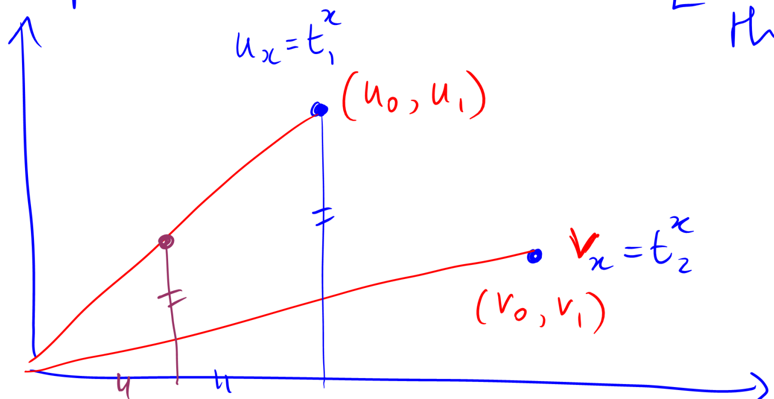
oops, complex, but you get the idea!

We have discovered that $\begin{cases} t_1^x \text{ is a solution for all } x \\ t_2^x \text{ " " " " " } x \end{cases}$

If these solutions are linearly independent, then they can be our basis! If so, then ALL solutions must be of the form $u_x = A t_1^x + B t_2^x$ for some constants $A$ & $B$.

Linearly independent?

[Presume $t_1, t_2 \in \mathbb{R}$, they will be in this course]




We need these two solutions to be on different lines (not parallel to each other) for them to span the whole 2-d space. If so, we can use them as basis vectors.

According to handout, check $\det\begin{vmatrix} u_0 & v_0 \\ u_1 & v_1 \end{vmatrix} \neq 0$ ??

If $\det\begin{pmatrix} u_0 & v_0 \\ u_1 & v_1 \end{pmatrix} = 0 \Rightarrow u_0 v_1 - \underbrace{u_1 v_0} = 0$

$$\Rightarrow \frac{u_0}{u_1} \cdot \frac{v_1}{v_0} - 1 = 0$$

$$\Rightarrow \frac{u_0}{u_1} = \frac{v_0}{v_1}$$

$\Rightarrow$ parallel.

If $\det\begin{pmatrix} u_0 & v_0 \\ u_1 & v_1 \end{pmatrix} \neq 0 \Rightarrow \{u_x\}$ and $\{v_x\}$ are NOT parallel solutions.

So they span the whole solution space, and ALL solns must be of the form

$w_x = A\, u_x + B\, v_x$ for some scalars $A$ and $B$.

i.e. $w_x = A\,(t_1^x) + B(t_2^x)$ for all $x$.

So we've discovered: if

(H) : $a u_{x+2} + b u_{x+1} + c u_x = 0$

then try to solve the characteristic eqn:

$$a t^2 + b t + c = 0 \qquad (*)$$

If this gives two real roots, $t_1$ and $t_2$ with $t_1 \neq t_2$, then the two solutions $\{t_1^x\}$ and $\{t_2^x\}$ can be used as basis vectors for the whole solution space, as long as they are linearly indept [check].

So ALL solutions are of the form

$$u_x = A t_1^x + B t_2^x \quad \text{for some } A \text{ and } B.$$

Use boundary conditions to find A and B.

If (*) has just one repeated root, need to make a different lucky guess: see handout

— this happened in §2.7-ish, Symmetric Gamblers Ruin: general soln was $u_x = A + Bx$.

(*) was $(t-1)^2 = 0$ ??

$\Rightarrow t=1, t=1$

What about the original equation (E) ?

$$m_{x+2} - m_{x+1} + m_x = c_x \quad . \quad (E)$$

Suppose I could find just ONE solution of (E). $\{m_x\}$.
It might not be the one I want.
Suppose the solution I want is called $\{y_x\}$.

Then $\{m_x\}$ satisfies (E), so $m_{x+2} - m_{x+1} + m_x = c_x$ (1)
And $\{y_x\}$ " " so $y_{x+2} - y_{x+1} + y_x = c_x$ (2)

(1) − (2) : $(m_{x+2} - y_{x+2}) - (m_{x+1} - y_{x+1}) + (m_x - y_x) = 0$

and THAT is my eqn (H) !!!

So $\{m_x - y_x\}$ is a solution of (H).

known by lucky guess

unknown & wanted

already solved

So $m_x - y_x = A t_1^x + B t_2^x$ for some $A, B$

$$\therefore \quad y_x = m_x + \underbrace{A' t_1^x + B' t_2^x}_{\text{general soln to (H)}} \quad \text{for some } A', B'.$$

particular solution e.g. $m_x = \text{const}$

general solution to the original equation (E)

Find $A'$ and $B'$ by boundary conditions. ☺

## Overall Solution to (E):

[ lay out your answers like this, INCLUDING titles ]

1) State original eqn (E) and boundary conditions.

Equation (E)

2) Homogeneous Equation (H)

State eqn (H)

No boundary conditions here, we've only got boundaries for (E)

Characteristic Equation:

State quadratic & solve for $t_1, t_2$

Check for linear independence:

$$\begin{vmatrix} u_0 & v_0 \\ u_1 & v_1 \end{vmatrix} \neq 0 \qquad \begin{array}{ll} u_0 = t_1^0 & u_1 = t_1^1 \\ v_0 = t_2^0 & v_1 = t_2^1 \end{array}$$

General solution for (H)

$u_x = A t_1^x + B t_2^x$ or whatever.

3) Particular Solution for (E):

try $m_x$ = constant $= c$ and solve for $c$

If not, try $m_x = cx$ & solve for $c$

→ see handout

4) General solution for (E):

$$m_x = \text{particular soln} + \text{general soln for (H)}$$

$$= \text{e.g. } cx + A t_1^x + B t_2^x$$

5) Boundaries for E $\Rightarrow$ solve for A & B.

6) Final solution of E : state.