$X_1 \quad X_2 \quad X_3 \quad X_4 \qquad \boxed{X_5}$

$\mathbb{E}(X_5 \mid \text{all previous}) = X_4$

Defn $\{X_t\}_{t=0,1,2,\ldots}$ is a martingale if

1) $\mathbb{E}(X_{t+1} \mid X_t, \underbrace{X_{t-1}, \ldots, X_0}_{\text{these don't matter}}) = X_t$ ✓

given current state X_t, we expect no change on average for X_{t+1}

"fair game" ie. don't expect a profit or loss from this state to the next state;

"break even" on average.

2) $\mathbb{E}(|X_t|) < \infty$ for all X_t. [can't spiral out of control]

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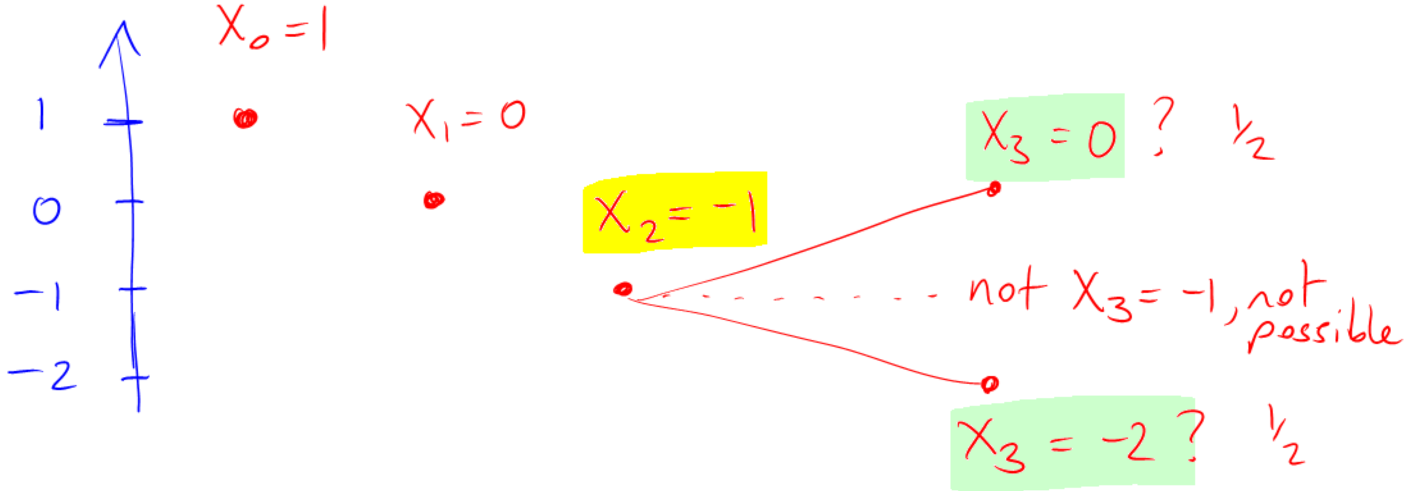
e.g. Random walk $\boxed{x-1} \xleftarrow{1/2} \boxed{x} \xrightarrow{1/2} \boxed{x+1}$

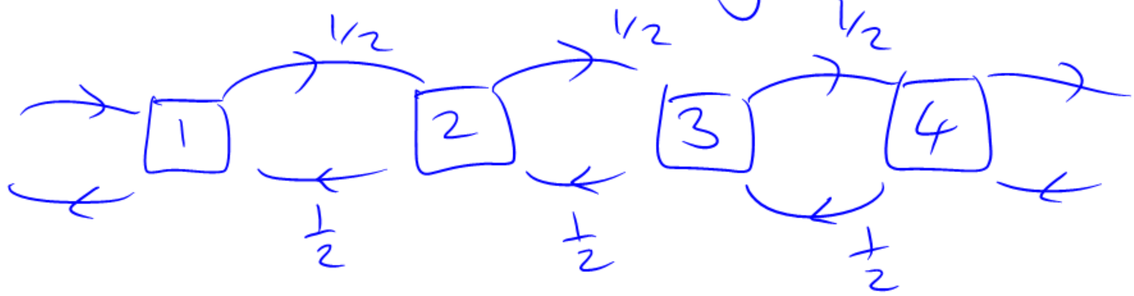


$\mathbb{E}(X_3 \mid X_2) = \frac{1}{2} * 0 + \frac{1}{2}(-2)$
$= -1 = X_2$

$\{X_t\}_{t \geq 0}$ is a martingale if $\mathbb{E}(|X_t|) < \infty$ for all t, and if $\mathbb{E}(X_{t+1} \mid X_t, \ldots, X_0) = X_t$.

Sometimes we have a sequence $S_0, S_1, S_2, \ldots$ that might not itself be a martingale, but it can be used to construct another martingale.

e.g.



Let $S_0, S_1, S_2, \ldots$ be the state we're in at time t. (S_t = which box we're in @ t)

Define a new sequence $M_0, M_1, M_2, \ldots$ such that

$$M_t = S_t^2 - t \text{ for all } t.$$

(M_t is not sthg we can put on the diagram – not a Markov chain)

e.g.

	S_0	S_1	S_2	S_3
	2	3	2	1

	M_0	M_1	M_2	M_3	
$\Rightarrow$	$2^2 - 0$	$3^2 - 1$	$2^2 - 2$	$1^2 - 3$	
	$= 4$	$= 8$	$= 2$	$= -2$	etc ....

We say $M_0, M_1, M_2, \ldots$ is a martingale with respect to $S_0, S_1, S_2, \ldots$ if:

i) $\mathbb{E}(|M_t|) < \infty$ for all t, and

ii) $\mathbb{E}(M_{t+1} \mid S_t, S_{t-1}, \ldots, S_0) = M_t$

Example (previous example)

$S_0, S_1, S_2, \ldots$ symmetric random walk as before

$M_t = S_t^2 - t$

$\mathbb{E}(M_{t+1} \mid S_t, \ldots, S_0) = \; ?$

$$S_{t+1} = \begin{cases} S_t + 1 & \text{w.p. } 1/2 \\ S_t - 1 & \text{w.p. } 1/2 \end{cases}$$

so $M_{t+1} = S_{t+1}^2 - (t+1)$

$$\therefore M_{t+1} = \begin{cases} S_t^2 + 2S_t + 1 - t - 1 & \tfrac{1}{2} \\ S_t^2 - 2S_t + 1 - t - 1 & \tfrac{1}{2} \end{cases} \qquad \text{prob}$$

$\therefore \mathbb{E}(M_{t+1} \mid S_t, \ldots, S_0) = S_t^2 - t = M_t$

So $M_t = S_t^2 - t$ forms a martingale with respect to $S_0, S_1, S_2, \ldots$

How Martingales are Used

Key principle: "you can't make money on a fair game"
on average

Let $S_0, S_1, \ldots$ be a martingale.

Idea: if $\mathbb{E}(S_1 \mid S_0 = x) = x$

Then $\mathbb{E}(S_2 \mid S_1, S_0) = S_1$

$$\therefore \quad \mathbb{E}_{S_1}\{\mathbb{E}(S_2 \mid S_1, S_0)\} = \mathbb{E}_{S_1}\{S_1 \mid S_0\} = x$$

$$\vdots$$

$$\Rightarrow \quad \mathbb{E}(S_t \mid S_0 = x) = x \quad \text{for any } t.$$

Question: does this still apply if t is random?

i.e. if I go a RANDOM # steps, T, into the future, does it still apply that

$$\longrightarrow \quad \mathbb{E}(S_T \mid S_0 = x) = x \quad ?$$

Answer: we need the Optional Stopping Theorem to tell us. Sometimes answer = yes, sometimes no.

Defn: random variable T is a stopping time for $S_0, S_1, S_2, \ldots$ if we always know in the present whether or not it's time to stop.

i.e. the event $\{T = t\}$ [stop at time t] depends only on $S_0, S_1, \ldots, S_t$, not on any future observations $S_{t+1}, S_{t+2}, \ldots$.

e.g. Gamblers Ruin.

T = FIRST time at which $S_t = 0$ or $S_t = N$.

e.g.

S_0	S_1	S_2	S_3	S_4
2	3	2	1	0

$T=4$ know to stop b/c $S_4 = 0$ and none of previous S_i's were 0 or N.

$\rightarrow$ don't need to know any future S_t's.

$\therefore$ This T is a stopping time.

$$\mathbb{E}(S_t \mid S_0 = x) = x$$

i.e. missing steps $S_1, S_2, \ldots, S_{t-1}$

yes

$$\mathbb{E}(S_2 \mid S_0 = x) = \sum_{s_2} s_2 \, \mathbb{P}(S_2 = s_2 \mid S_0 = x)$$

$$= \sum_{s_2} \sum_{s_1} s_2 \, \mathbb{P}(S_2 = s_2 \mid S_1 = s_1, S_0 = x) * \mathbb{P}(S_1 = s_1 \mid S_0 = x)$$

Optional Stopping Theorem (O.S.T.): [on your Ass B]

Let $S_0, S_1, \ldots$ be a martingale and T be a stopping time.

If: i) $\mathbb{E}(T) < \infty$

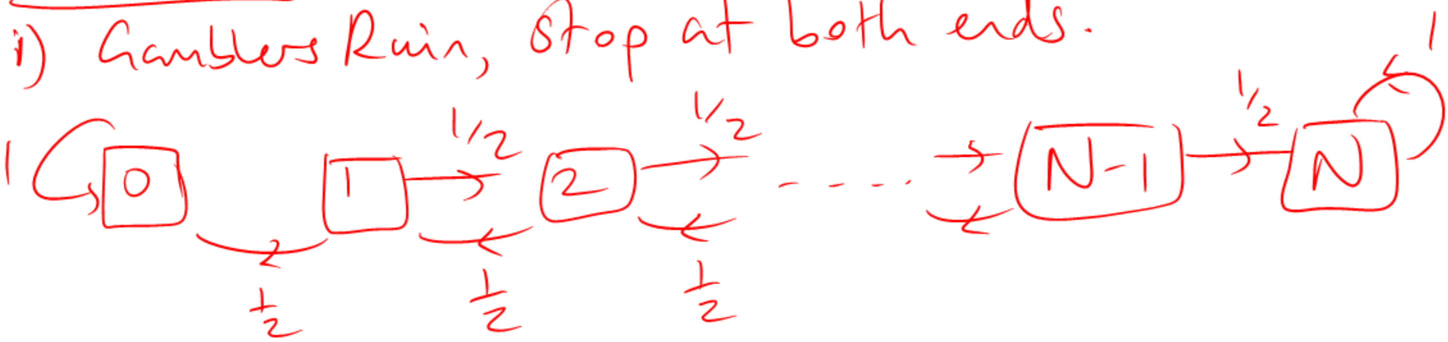
and ii) there exists a constant c such that
$$\mathbb{E}\{\,|S_{t+1} - S_t|\,\} \leq c \text{ for all } t,$$

then $\mathbb{E}(S_T) = S_0$.

[brief version: see Ass B p. 2 for full version]

Examples:

i) Gamblers Ruin, stop at both ends.



Shown the stopped process on the diagram.

T = time reach 0 or N for first time.

Does it meet conditions of O.S.T. ?

Yes. $\mathbb{E}(T) < \infty$ [obvious but Ass says you can assume]

$\mathbb{E}\{\,|S_{t+1} - S_t|\,\} = 1$ because $|S_{t+1} - S_t| = 1$ always

So condition (ii) also satisfied.

So OST $\Rightarrow$ $\mathbb{E}(S_T) = S_0$.

Suppose $S_0 = x$ (like Ch 2, §2.7 325 notes)

$$S_T = \begin{cases} 0 & \text{w.p. } 1-w \quad (\text{prob lose}) \\ N & \text{w.p. } w \quad (\text{prob we win}) \end{cases}$$

So $\mathbb{E}(S_T) = 0 * (1-w) + Nw = Nw$

OST $\Rightarrow$ $\mathbb{E}(S_T) = x$ finish = start on average.

So $Nw = x$

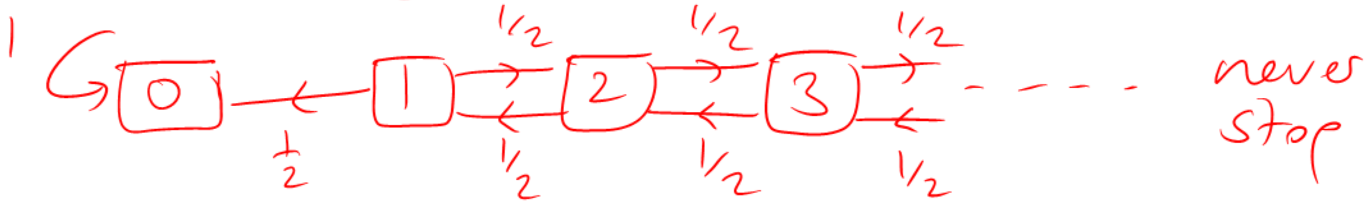
$$\underline{\underline{w = \frac{x}{N}}}$$

Wow! found the winning probability with almost no work!!

Example 2: when it doesn't work

Only stop at ONE end (0)

Foolish gambler, only stops when broke.



Start at x. T = time at which first reach 0.

Is it true that $\mathbb{E}(S_T) = x$??

No! $\mathbb{E}(S_T) = 0$ obviously $\neq x$

What's wrong? i) $\mathbb{E}(T) < \infty$? No!! See 325. Why it fails.

When you do Q3, consider why O.S.T. fails for that diag. if $q > p$ (so $\mathbb{E}T < \infty$) but we only stop at 0.

Stopped processes

e.g. Gamblers Ruin stopped at 0 or N.

Say $S_0 = 3$.

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
unstopped process	3	4	3	2	1	2	1	0	−1	−2
stopped process	3	4	3	2	1	2	1	0	0	0

Unstopped process : $\{S_t\}$ or $\{S_n\}$

Stopped process : $\{S_{T \wedge t}\}$ or $\{S_{T \wedge n}\}$

or $\{S_{\min(T,t)}\}$

If $S_0, S_1, S_2, \ldots$ is a martingale & T is a stopping time, then $\{S_{T \wedge t}\}$ is also a martingale.

[Obvious but useful and occasionally tricky — e.g. the transition diagram usually shows the STOPPED process.]

Big part of answers is checking conditions:

– martingale needs $\mathbb{E}(|M_t|) < \infty$

Use triangle inequality,

e.g. Gamblers Ruin $S_0, S_1, \ldots$

$$S_t = x + \underbrace{Y_1 + Y_2 + \cdots + Y_t}_{\text{all these are } \pm 1}$$

So $|S_t| \leq |x| + |Y_1| + |Y_2| + \cdots + |Y_t|$

$= x + 1 + 1 + \cdots + 1$

$= x + t$ (finite)

$\therefore \mathbb{E}(|S_t|) < \infty$

<u>OST</u> : can assume $\mathbb{E}(T) < \infty$ for 721

but must check $\mathbb{E}(|M_{n+1} - M_n|) \leq c$ for all n.

Due Mon 17th 4pm
721 box.