Def. \( \{ X_t \}_{t=0,1,2,...} \) is a martingale if

1) \( \mathbb{E}(X_{t+1} | X_t, X_{t-1}, ..., X_0) = X_t \)  
   these don't matter
   given current state \( X_t \), we expect no change
   on average for \( X_{t+1} \)
   "fair game" i.e. don't expect a profit or loss
   from this state to the next state;
   "break even" on average.

2) \( \mathbb{E}(|X_{t+1}|) < \infty \) for all \( X_t \).  
   \(
   \left[ \begin{array}{c}
   \text{can't spiral} \\
   \text{out of control}
   \end{array}
   \right]
   \)

E.g. Random walk \((X-1) \xleftarrow{\text{-}} X \xrightarrow{\text{1/2}} X+1\)

\( X_0 = 1 \)

\( X_1 = 0 \)

\( X_2 = -1 \)

\( X_3 = 0 \) ? \( \frac{1}{2} \)

not \( X_3 = -1 \), not possible

\( X_3 = -2 \) ? \( \frac{1}{2} \)

\( \mathbb{E}(X_3 | X_2) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot (-2) = -1 = X_2 \)
{X_t}_{t \geq 0} is a martingale if $E(1|X_t|) < \infty$
for all $t$, and if $E(X_{t+1} | X_t, \ldots, X_0) = X_t$.

Sometimes we have a sequence $S_0, S_1, S_2, \ldots$ that might not itself be a martingale, but it can be used to construct another martingale.

E.g.

Let $S_0, S_1, S_2, \ldots$ be the state we're in at time $t$. ($S_t$ = which box we're in @ $t$)

Define a new sequence $M_0, M_1, M_2, \ldots$ such that $M_t = S_t^2 - t$ for all $t$.

($M_t$ is not itself, we can put on the diagram = not a Markov chain)

e.g. $S_0, S_1, S_2, S_3$

2 3 2 1

$\Rightarrow M_0, M_1, M_2, M_3$

$2^2 - 0$ $3^2 - 1$ $2^2 - 2$ $1^2 - 3$

$= 4$ $= 8$ $= -2$ $= -2$

c.t.c. ...
We say $M_0, M_1, M_2, \ldots$ is a martingale with respect to $S_0, S_1, S_2, \ldots$ if:

i) $E(\ln |M_t|) < \infty$ for all $t$, and

ii) $E(M_{t+1} \mid S_t, S_{t-1}, \ldots, S_0) = M_t$

---

**Example (previous example)**

$S_0, S_1, S_2, \ldots$ symmetric random walk as before

$M_t = S_t^2 - t$

$E(M_{t+1} \mid S_t, \ldots, S_0) = \ ?$

$S_{t+1} = \begin{cases} S_t + 1 & \text{w.p. } \frac{1}{2} \\ S_t - 1 & \text{w.p. } \frac{1}{2} \end{cases}$

So $M_{t+1} = S_{t+1}^2 - (t+1)$

$M_{t+1} = \begin{cases} S_t^2 + 2S_t + 1 - t - 1 & \frac{1}{2} \\ S_t^2 - 2S_t + 1 - t - 1 & \frac{1}{2} \end{cases}$

$\therefore E(M_{t+1} \mid S_t, \ldots, S_0) = S_t^2 - t = M_t$

So $M_t = S_t^2 - t$ forms a martingale with respect to $S_0, S_1, S_2, \ldots$. 
How Martingales are Used

Key principle: "you can't make money on a fair game" on average

Let $S_0, S_1, \ldots$ be a martingale. Then

$E(S_0) \equiv E(S_1) \equiv E(S_2) \equiv \ldots \equiv \ldots$

$E(S_0) \equiv E(S_1) \equiv E(S_2) \equiv \ldots \equiv \ldots$

Then $E(S_2 | S_1, S_0) = S_1$

$E(S_3 | S_2, S_1, S_0) = E(S_4 | S_3, S_2, S_1, S_0) = \ldots = \ldots$

$E(S_t | S_0 = x) = x$ for any $t$.

Question: does this still apply if $t$ is random? i.e. if I go a RANDOM $T$ steps, $T$, into the future, does it still apply that

$E(S_T | S_0 = x) = x$ ?

Answer: we need the Optional Stopping Theorem to tell us. Sometimes answer = yes, sometimes no.

Defn: random variable $T$ is a stopping time for $S_0, S_1, S_2, \ldots$ if we always know in the present whether or not it's time to stop.

i.e. the event $T = t$ depends only on $S_0, S_1, \ldots, S_t$, not on any future observations $S_{t+1}, S_{t+2}, \ldots$
e.g. Gamblers Ruin.

\[ \begin{align*}
T &= \text{first time at which } S_t = 0 \text{ or } S_t = N. \\
\text{e.g. } S_0, S_1, S_2, S_3, S_4 \\
2, 3, 2, 1, 0 \\
T = 4 \text{ because } S_4 = 0 \text{ and none of the previous } S_i's \text{ were } 0 \text{ or } N. \\
\rightarrow \text{don't need to know any future } S_t's. \\
\text{This } T \text{ is a stopping time.}
\end{align*} \]

\[
\mathbb{E}(S_t \mid S_0 = x) = x
\]

i.e. missing steps \( S_1, S_2, \ldots, S_{t-1} \)

yes

\[
\mathbb{E}(S_2 \mid S_0 = x) = \sum_{s_2} s_2 \Pr(S_2 = s_2 \mid S_0 = x)
\]

\[
= \sum_{s_2} \sum_{s_1} s_2 \Pr(S_2 = s_2 \mid S_1 = s_1, S_0 = x) \\
\times \Pr(S_1 = s_1 \mid S_0 = x)
\]
Optional Stopping Theorem: [on your Ass B]

Let \( S_0, S_1, \ldots \) be a martingale and \( T \) be a stopping time.

If:
1) \( \mathbb{E}(T) < \infty \)
2) there exists a constant \( c \) such that
\[
\mathbb{E}[|S_{t+1} - S_t|^2] \leq c \quad \text{for all } t,
\]
then \( \mathbb{E}(S_T) = S_0 \). [brief version: see Ass B p. 2 for full version]

Examples:
1) Gambler's Ruin, stop at both ends.
\[
\begin{array}{cccccc}
0 & 1 & 2 & \cdots & (N-1) & N \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array}
\]

shown the stopped process on the diagram.

\( T = \) time reach 0 or \( N \) for first time.

Does it meet conditions of O.S.T.?

Yes. \( \mathbb{E}(T) < \infty \) [obvious but Ass says you can assume]

\[ \mathbb{E}[|S_{t+1} - S_t|^2] = 1 \] because \( |S_{t+1} - S_t| = 1 \) always

So condition (ii) also satisfied.
So $\text{OST} \Rightarrow \mathbb{E}(S_T) = S_0$.

Suppose $S_0 = x$ (like Ch 2, § 2.7 325 notes)

\[ S_T = \begin{cases} 0 \text{ w.p. } 1-w \text{ (prob lose)} \\ N \text{ w.p. } w \text{ (prob win)} \end{cases} \]

So $\mathbb{E}(S_T) = 0 \times (1-w) + Nw = Nw$

$\text{OST} \Rightarrow \mathbb{E}(S_T) = x$  \hspace{1cm} \text{finish} = \text{start on average.}$

So $Nw = x$  

\[ w = \frac{x}{N} \]

Wow! Found the winning probability with almost no work!!

\underline{Example 2: when it doesn’t work}

Only stop at ONE end (0)

Foolish gambler, only stops when broke.

\[ S_0 \xrightarrow{\frac{1}{2}} 1 \xrightarrow{\frac{1}{2}} 2 \xrightarrow{\frac{1}{2}} 3 \xrightarrow{\frac{1}{2}} \cdots \text{ never stop} \]

Start at $x$. $T =$ time at which first reach 0.

Is it true that $\mathbb{E}(S_T) = x$??

No! $\mathbb{E}(S_T) = 0$ obviously $\neq x$

What’s wrong? 1) $\mathbb{E}(T) < \infty$? No!! See 325. Why it fails
When you do Q3, consider why O.S.T. fails for that diag. if $q > p$ (so $\mathbb{E}T < \infty$) but we only stop at 0.

---

**Stopped processes**

E.g. Gamblers Ruin stopped at 0 or N.
Say $S_0 = 3$.

\[
\begin{align*}
S_0 & \rightarrow 3 \\
S_1 & \rightarrow 4 \\
S_2 & \rightarrow 3 \\
S_3 & \rightarrow 2 \\
S_4 & \rightarrow 1 \\
S_5 & \rightarrow 2 \\
S_6 & \rightarrow 1 \\
S_7 & \rightarrow 0 \\
S_8 & \rightarrow 1 \\
S_9 & \rightarrow 0 \\
\end{align*}
\]

Unstopped process: $\{S_t\}$ or $\{S_n\}$

Stopped process: $\{S_{\min(T,t)}\}$ or $\{S_{T \wedge t}\}$ or $\{S_{\min(T,t)}\}$

If $S_0, S_1, S_2, \ldots$ is a martingale & $T$ is a stopping time, then $\{S_{T \wedge t}\}$ is also a martingale.

[Obvious but useful and occasionally tricky — e.g. the transition diagram usually shows the STOPPED process.]
Big part of answer is checking conditions:
- Martingale needs \( \mathbb{E}(|M_{t+1}|) < \infty \)

Use triangle inequality,
e.g., Gambler's Ruin \( S_0, S_1, \ldots \)

\[
S_t = x + Y_1 + Y_2 + \ldots + Y_t
\]

all these are \( \pm 1 \)

So \( |S_t| \leq |x| + |Y_1| + |Y_2| + \ldots + |Y_t| \)

\[= x + |1 + 1 + \ldots + 1| \]
\[= x + t \quad \text{(finite)} \]

\[
\therefore \mathbb{E}(|S_t|) < \infty
\]

OST: can assume \( \mathbb{E}(T) < \infty \) for 721
but must check \( \mathbb{E}(|M_{t+1} - M_t|) \leq c \)
for all \( t, n \).

Due Mon 17th 4pm 721 box.