Why do we worry about Stopped Processes?

Quick answer: To use O.S.T. we need to show that

$$\mathbb{E}\left(|M_{t+1} - M_t| \right) \leq c \quad \text{for some constant } c. \quad (*)$$

(You need to include this in your answers.)

For some martingales (e.g. Q2), this is satisfied by the stopped process $\{M_{T \wedge t}\}$, but not by the unstopped process $\{M_t\}$.

Line of argument:

If $\{M_t\}$ is a martingale with respect to $\{S_t\}$,
and if T is a stopping time " " " $\{S_t\}$,
then $\{M_{T \wedge t}\}$ is also a martingale w.r.t. $\{S_t\}$. (p.2 Ass B)

So if we can show that $\mathbb{E}\left(|M_{t+1} - M_t| \right) \leq c$ for the STOPPED process (ie. for $t < T$), then we can use the O.S.T. with the STOPPED process to deduce that

$$\mathbb{E}(M_T) = \mathbb{E}(M_0).$$

Note: to be more precise, the condition should say

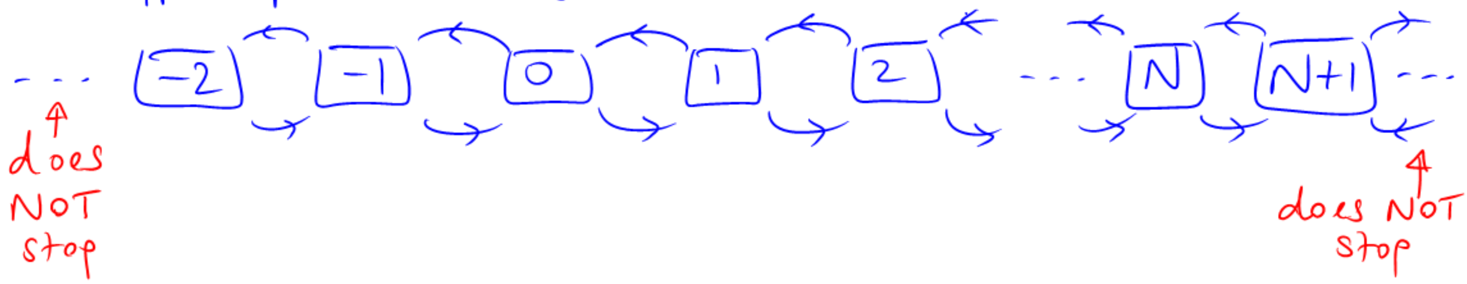
$$\mathbb{E}\Big(\underbrace{|M_{t+1} - M_t|}_{\text{absolute value}} \; \underbrace{\Big|\; M_t}_{\text{given } M_t} \Big) \leq c \quad \text{for all } t.$$

The point is that the step sizes need to be bounded by a single constant c.

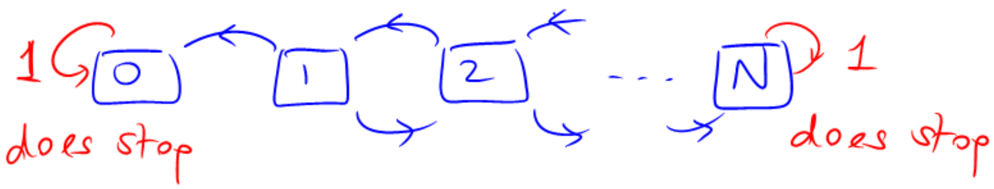
e.g. see book Grimmett & Stirzaker p.492 Thm 9.

Example from Q2:

Unstopped process $\{S_t\}$ looks like this: (1)



Stopped process $\{S_{T \wedge t}\}$ looks like this: (2)



Now consider $\mathbb{E}(|S_{t+1} - S_t|)$:

$S_{t+1} - S_t = \pm 1$ in EITHER stopped process OR unstopped process

So $\mathbb{E}(|S_{t+1} - S_t|) = 1$ always, so condition (*) of O.S.T. is always satisfied (using $c = 1$) and it doesn't matter whether we use the stopped or unstopped process.

BUT... now consider $M_t = S_t^2 - t$:

$$|M_{t+1} - M_t| = 2|S_t| \text{ for all } t$$ [you should show this in your answers]

There is NO constant c that can bound this for all t in the UNSTOPPED process. The unstopped process can continue without bound: see diagram (1) above.

But in the stopped process, $|S_t| \leq N$ for all t.

So $\mathbb{E}(|M_{t+1} - M_t|) \leq 2N$ for all t, so the STOPPED process satisfies condition (*) of O.S.T. using $c = 2N$.

Finally, when you have decided you can use O.S.T. in Q 2b,

use $$\mathbb{E}(M_T) = \mathbb{E}(S_T^2 - T)$$
$$= \mathbb{E}(S_T^2) - \mathbb{E}T$$

where $$S_T^2 = \begin{cases} ? & \text{with prob. } ? \\ ? & \text{w.p. } ? \end{cases}$$

Incorporate the O.S.T. and solve for $\mathbb{E}T$.