STATS 721 (2012) FINAL EXAM SOLUTIONS:

1) a) $E[X_{t+1}^2 \mid X_0, \ldots X_t, Y_0, \ldots Y_t] =$

$$\tfrac{2}{8}(X_t-2)^2 + \tfrac{2}{8}(X_t-1)^2 + \tfrac{2}{8}(X_t+1)^2 + \tfrac{2}{8}(X_t+2)^2$$

$$= X_t^2 + \tfrac{1}{4}(4+1+1+4)$$

$$= X_t^2 + \tfrac{5}{2}.$$

Similarly,

$$E[Y_{t+1}^2 \mid X_0, \ldots X_t, Y_0, \ldots Y_t] = Y_t^2 + \tfrac{5}{2}.$$

So

$$E[R_{t+1}^2 \mid \ldots] = R_t^2 + 5.$$

$$E[M_{t+1} \mid \ldots] = R_t^2 + 5 - \beta(t+1)$$

$$= M_t + 5 - \beta$$

and thus $\beta = 5$ is the only possible value.

b) Minimum value is $\begin{cases} 0, & \text{if } t \text{ is even (return to start)} \\ 5, & \text{if } t \text{ is odd (one move away from start)} \end{cases}$

Maximum is $5t^2$ (all t moves in the same direction)

Expected value is (by the martingale property)

$$E[R_t^2] = E[M_t + 5t] = E[M_0] + 5t = 5t.$$

c) We know when to stop : the event $\{T = t\}$ is determined by $(X_0, \dots X_t, Y_0, \dots Y_t)$.

d) The process will stop whenever there are $m = \lceil \frac{2r}{\sqrt{5}} \rceil$ consecutive moves towards the boundary circle.
The probability that this happens in any m consecutive moves is $\left(\frac{1}{8}\right)^m > 0$. So T/m is dominated by a geometric random variable.

e) Optional stopping theorem.
Let $M'_t = M_{T \wedge t}$. We need to check

$$E[\,|M'_{t+1} - M'_t|\,] \quad \text{bounded};$$

this holds because $|M'_{t+1} - M'_t| = |R^2_{T\wedge(t+1)} - R^2_{T\wedge t} - 5|$

$$\leq 2r^2 + 5 < \infty.$$

(Also $E[T] < \infty$ as noted in d)).
O.S.T. gives us

$$E[M_T] = E[M'_T] = E[M'_0] = 0.$$

$$\therefore \quad E[R_T^2] - 5.E[T] = 0$$

$$\text{so} \quad E[T] = \frac{E[R_T^2]}{5} \geq \frac{r^2}{5}.$$

~~3) ... $u_k = 1 + \frac{3}{4}u_{k+1} + \frac{1}{4}u_{...}$~~

3) ~~(e)~~ The generating-function calculation in (c), (d) are not valid here because they rely on translation-invariance.

For (c), the probability of never reaching state 1 is the same (b); adding a boundary beyond state 1 cannot change this.

For (d), let $p_k = P(\text{reach state } 0)$ starting from k.

Then $p_0 = 1$, ~~and~~ $p_{100} = 0$, and
$p_k = \frac{3}{4} p_{k+1} + \frac{1}{4} p_{k-1}$.

Solve this difference equation.
The probability we want ~~is~~ is $1 - \left(\frac{3}{4} p_1 + \frac{1}{4} p_{-1}\right)$.