1.

(a) Let

$$T' = \min\{t : t \text{ is a multiple of } N, \text{ with the last } N \text{ steps before } t \text{ all being downward}\}$$

Then $T \le T'$, because N downward steps in a row will certainly bring the process to 0. Also, $(T'/N) - 1$ has a geometric distribution (with parameter 2^{-N}), and so has finite mean. Hence $E[T'] < \infty$, and so $E[T] < \infty$.

(b)

$$t_k = 1 + \frac{t_{k-1} + t_{k+1}}{2},$$

with boundary conditions $t_0 = t_N = 0$.

(c) First find the solution to the homogeneous version of the equation:

$$t_{k+1} - 2t_k + t_{k-1} = 0.$$

The characteristic equation

$$\lambda^2 - 2\lambda + 1 = 0$$

has a single root $\lambda = 1$. So the general solution to the homogeneous equation is

$$t_k = Ak + B$$

for arbitrary constants A and B. Now look for a particular solution to the original equation. Try

$$t_k = ck^2.$$

Substituting this into the equation leads to $c = -1$. So the general solution to the original equation is

$$t_k = -k^2 + Ak + B.$$

Using the two boundary conditions to solve for A and B yields $A = N$ and $B = 0$. So the final solution is

$$t_k = k(N - k).$$
