STATS 721 Solutions to 721 Q2 & 4 Semester 2, 2011

Q2a) We need to show $\mathbb{E}\{|X_t|\} < \infty$ for all t, (i)

and $\mathbb{E}(X_{t+1} \mid X_t, X_{t-1}, \dots, X_0) = X_t$. (ii)

i) Clearly, $0 \le X_t \le N$ for all t, so $\underline{\underline{\mathbb{E}\{|X_t|\} < \infty}}$.

ii) $[X_{t+1} \mid X_t] \sim \text{Binomial}\left(N, \frac{X_t}{N}\right)$

so $\mathbb{E}(X_{t+1} \mid X_t, \dots, X_0) = \mathbb{E}(X_{t+1} \mid X_t)$ (Markov Property)

$= N * \frac{X_t}{N}$ (Binomial mean)

$\underline{\underline{= X_t \text{ as required.}}}$

Thus $X_0, X_1, \dots$ is a martingale.

b) $T = \min\{t : X_t = 0 \text{ or } X_t = N\}$.

So T depends only on events visible up to and including time T, not on any future events.

This is the requirement for T to be a stopping time w.r.t. $\{X_t\}$.

c) For the Optional Stopping Theorem, we need:

i) $\mathbb{E}T < \infty$ (given in Q)

ii) There exists some constant c such that

$\mathbb{E}(|X_{t+1} - X_t|) \le c$ for all t.

This is satisfied putting $c = N$, because

$0 \le X_t \le N$ for all t.

So we can apply the Optional Stopping Theorem.

2c cont.) By O.S.T. and because T is a stopping time w.r.t. $\{X_t\}$,

$\underline{\mathbb{E}(X_T) = \mathbb{E}(X_0) = x}$ in this case. (a)

Now let $p = \mathbb{P}(\text{eventually fixed at state } N)$.

By definition of T,

$$X_T = \begin{cases} N & \text{with prob. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

So $\underline{\mathbb{E}(X_T) = pN + (1-p)*0 = pN}$. (b)

Equating (a) and (b) gives $x = pN$

$\Rightarrow \underline{\underline{p = \frac{x}{N}}}$.

This is the probability the population is eventually fixed for allele A, ie. chain stops in state N.

Q(4a) First-step analysis: $h_x = \frac{2}{3} h_{x-1} + \frac{1}{3} h_{x+1}$ $x = 1, \dots, N-1$

So $\underline{h_{x+1} - 3h_x + 2h_{x-1} = 0}$ (*)

Valid for $x = 1, \dots, N-1$.

$\underline{\text{Boundaries:}}$ $\underline{\underline{h_0 = 0 \quad h_N = 1.}}$

The eqn (*) is homogeneous, so we can proceed.

$\underline{\text{Characteristic equation:}}$ $t^2 - 3t + 2 = 0$

$(t-1)(t-2) = 0$

$\Rightarrow \underline{\underline{t = 1, \ t = 2}}$ distinct real roots.

4a cont.) Deduce solutions to (*) of form $\begin{cases} h_x = 1^x = 1 \\ h_x = 2^x. \end{cases}$

<u>Check for linear independence:</u>

$$\det \begin{pmatrix} 1^0 & 2^0 \\ 1^1 & 2^1 \end{pmatrix} = 2-1 = 1 \neq 0.$$

Thus (*) has two linearly independent solutions which form a basis for the 2-dimensional solution space.

$\therefore$ <u>General solution of (*):</u>

$$\underline{h_x = A + B\,2^x \quad \text{for } x = 0, 1, 2, \dots, N.} \qquad (**)$$

A, B constants to be found.

<u>Boundary conditions:</u>

$$h_0 = A + B = 0 \qquad (1)$$
$$h_N = A + B2^N = 1 \qquad (2)$$

(2) − (1): $\quad B(2^N - 1) = 1$

$$\therefore \quad \underline{\underline{B = \frac{1}{2^N - 1}, \quad A = -B.}}$$

<u>Final solution to (*):</u>

$$h_x = B(2^x - 1)$$

$$\therefore \quad \underline{\underline{h_x = \frac{2^x - 1}{2^N - 1} \quad \text{for } x = 0, 1, \dots, N}}$$

as stated.

4b) First-step analysis:

$$m_x = 1 + \tfrac{1}{3} m_{x+1} + \tfrac{2}{3} m_{x-1} \qquad x = 1, \dots, N-1$$

$$\Rightarrow \quad \underline{m_{x+1} - 3m_x + 2m_{x-1} = -3} \qquad (E)$$

for $x = 1, \dots, N-1$.

<u>Boundaries:</u> $\quad \underline{m_0 = m_N = 0}$. $\quad$ (given).

This is an inhomogeneous eqn, but we have already solved the corresponding homogeneous eqn in (a).

<u>Homogeneous equation:</u>

(H) $\quad u_{x+1} - 3u_x + 2u_{x-1} = 0 \quad$ same as (*) part (a)

<u>General solution to (H):</u>

$$\underline{u_x = C + D\,2^x \qquad x = 0, \dots, N.}$$

from (**) in (a).

C, D constants to be found.

<u>Particular solution to (E):</u>

Try $m_x = \theta x$:

$$\Rightarrow \quad \theta(x+1) - 3\theta x + 2\theta(x-1) = -3$$
$$\theta(\underbrace{x - 3x + 2x}_{0}) + \theta(1-2) = -3$$
$$\Rightarrow \quad \underline{\theta = 3.}$$

So $\underline{\underline{m_x = 3x}}$ is a particular solution to (E).

4b cont.) <u>General soln to E:</u>

$$\underline{m_x = C + D\,2^x + 3x} \qquad x = 0, \dots, N.$$

<u>Boundaries:</u> $\quad m_0 = C + D = 0 \qquad (1)$

$$m_N = C + D\,2^N + 3N = 0. \qquad (2)$$

$(2) - (1) \Rightarrow \quad D(2^N - 1) = -3N$

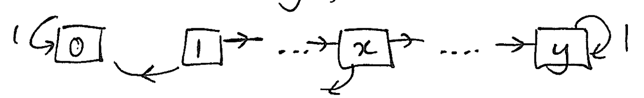
$$\therefore \quad D = \frac{-3N}{2^N - 1}, \quad C = -D.$$

<u>Final solution to E:</u>

$$m_x = \frac{-3N}{2^N - 1}(2^x - 1) + 3x$$

$$\Rightarrow \quad m_x = 3x - 3N\frac{(2^x - 1)}{2^N - 1} \qquad x = 0, \dots, N.$$

4c) For the event $\{Y \geq y\}$, consider the new transition diagram:



This is the same system as in (a) but with y replacing N.

So $\quad \mathbb{P}(Y \geq y) = \mathbb{P}(\text{chain hits } y \text{ before hitting } 0)$

$$= \underline{\frac{2^x - 1}{2^y - 1}} \quad \text{as long as } y \geq x.$$

For the event $\{Y = y\}$ we need the chain to hit y and <u>not</u> to hit $y+1$: $\quad \mathbb{P}(Y = y) = \mathbb{P}(Y \geq y) - \mathbb{P}(Y \geq y+1)$

for $y = x, \dots, N-1$.

4c cont.) Overall:

$$\mathbb{P}(Y = y) = \begin{cases} 0 & y = 0, \dots, x-1 \\ \dfrac{2^x - 1}{2^y - 1} - \dfrac{2^x - 1}{2^{y+1} - 1} & y = x, \dots, N-1 \\ \dfrac{2^x - 1}{2^N - 1} & y = N. \end{cases}$$

starting from state x.