1. We wish to find the probability of getting a run of at least 3 consecutive 0s or 1s in 15 tosses of a fair coin.

Set up a Markov chain similar to the one given in 325 Assignment 2, Question 3. Create a difference equation, and solve it to find the required probability. Also give a general formula for the probability of finding 3 consecutive 0s or 1s in \( t \) tosses of a fair coin. (10)

2. Consider a game following the transition diagram below. We say that the game is won if it finishes in state \( N \). Let \( S_0, S_1, S_2, \ldots \) be the process defined such that \( S_t \) is the state at time \( t \), and suppose we start with \( S_0 = x \). Assume that \( p < 0.5 \).

(a) Choose a suitable martingale with respect to \( S_0, S_1, S_2, \ldots \). Define the stopping time \( T \) to be the time at which the game finishes, either win or lose. Using the optional stopping theorem for bounded martingales, show that

\[
    w = \mathbb{P}(\text{game is won}) = \frac{x}{N},
\]

where \( x = S_0 \) is the start state. Your answer should include a proof that you have chosen a valid martingale. You may assume that \( \mathbb{P}(T < \infty) = 1 \). (5)

(b) Define a new process \( M_0, M_1, M_2, \ldots \), such that

\[
    M_t = S_t^2 - 2pt.
\]

Show that \( M_0, M_1, M_2, \ldots \) is a martingale with respect to \( S_0, S_1, S_2, \ldots \). (4)

(c) Recall that \( \mathbb{P}(T < \infty) = 1 \), and \( S_0 = x \). Using the optional stopping theorem for bounded martingales, together with a suitable choice of martingale from (a) or (b), find \( \mathbb{E}(T) \). You must clearly state which martingale you are using. (5)

Total: 24