STATS 721	Test	Semester 2, 2009

Answer **ALL QUESTIONS**. Marks are shown for each question.

Write your **name and ID number** at the top of your answer sheet.

Consider a random walk $S_0, S_1, S_2, \ldots$, where $S_0 = x$ and $S_t = x + \sum_{i=1}^{t} Y_i$, where $Y_1, Y_2, \ldots$ are independent random variables such that

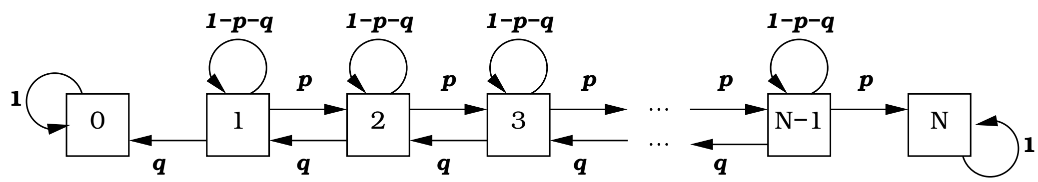
$$Y_i = \begin{cases} 1 & \text{with probability } p\,, \\ 0 & \text{with probability } 1 - p - q\,, \\ -1 & \text{with probability } q\,. \end{cases}$$

Suppose that the starting point, x, satisfies $0 < x < N$.

Let T be a random variable denoting the time at which the random walk first hits either state 0 or state N. The *stopped process* is the process that is stopped by T, defined as $S_{T \wedge t} = \{S_0, S_1, \ldots, S_T, S_T, S_T, \ldots\}$. You may assume that $\mathbb{P}(T < \infty) = 1$.

The diagram below is the transition diagram for the stopped process, $S_{T\wedge t}$. We say that the game is *won* if it stops in state N, and *lost* if it stops in state 0. The expected time taken for the game to finish is $\mathbb{E}(T)$. Our aim is to find $\mathbb{E}(T)$, which we will do (1) using second-order difference equations, and (2) using martingales.

Assume throughout that $p \neq q$.

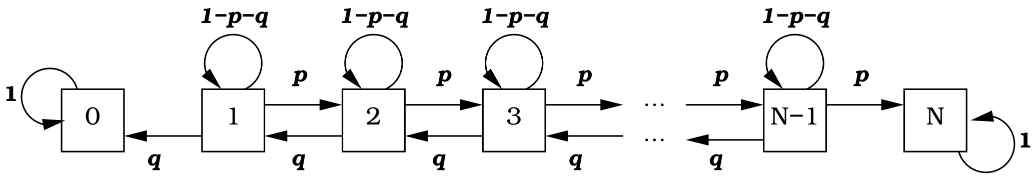


1. Let m_x be the expected number of steps for the game to finish, starting from state x, for $x = 0, 1, 2, \ldots, N$. (As in Stats 721 Assigment 1, the number of steps is the number of *arrows* traversed, not the number of *boxes* entered.)

   Formulate the second-order difference equation necessary to find m_x, and solve it to find m_x. Finish your answer by writing down an explicit expression for $\mathbb{E}(T)$, where T was the random variable defined above.

   [Hint: when solving the quadratic equation, factorize it by using an intelligent guess for one of the factors. Do **not** use the formula $(-b \pm \sqrt{b^2 - 4ac})/2a$.] (10)

The same diagram is shown again below for ease of reference. Remember that $p \neq q$.



2.(a) Using Stats 721 Assignment 2 as a guide, guess a suitable definition of a martingale with respect to $S_0, S_1, S_2, \ldots$. Prove that your process does indeed define a martingale. (5)

(b) Let w be the probability of winning the game (finishing in state N), starting from state x. Using the optional stopping theorem for bounded martingales, together with your answer for (a), show that

$$w = \mathbb{P}(\text{game is won}) = \frac{1 - \left(\frac{q}{p}\right)^x}{1 - \left(\frac{q}{p}\right)^N}.$$

(4)

(c) Using your answers to (a) and (b), find $\mathbb{E}(T)$. Say whether your answer is the same as you obtained in question 1, and whether or not it should be. (5)

Total: 24