STATS 721	Test	Semester 2, 2010

Answer **ALL QUESTIONS**. Marks are shown for each question.

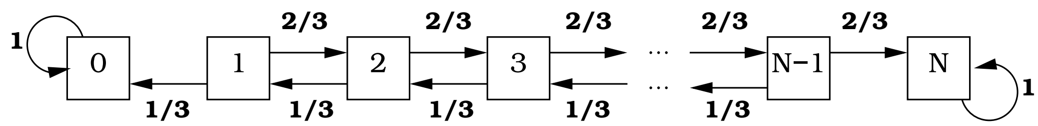
Write your **name and ID number** at the top of your answer sheet.

1. Consider a game consisting of a random walk $S_0, S_1, S_2, \ldots$. Let $S_0 = x$ and let $S_t = x + \sum_{i=1}^{t} Y_i$, where $Y_1, Y_2, \ldots$ are independent random variables such that

$$Y_i = \begin{cases} 1 & \text{with probability } 2/3\,, \\ -1 & \text{with probability } 1/3\,. \end{cases}$$

The starting point, x, satisfies $0 < x < N$.

The game finishes when it first hits either state 0 or state N. We say that the game is *won* if it stops in state N, and *lost* if it stops in state 0.



Let m_x be the expected number of steps for the game to finish, starting from state x, for $x = 0, 1, 2, \ldots, N$. (The number of steps is the number of *arrows* traversed.)

Formulate the second-order difference equation necessary to find m_x. Solve the equation to find m_x. (10)

2. Let $S_0, S_1, S_2, \ldots$ be a process such that $S_0 = x$, and for $t = 0, 1, 2, \ldots$,

$$S_{t+1} = \begin{cases} S_t + 2^t & \text{with probability } 1/2, \\ S_t - 2^t & \text{with probability } 1/2. \end{cases}$$

This process is an example of the **doubling strategy**, corresponding to a game where a gambler tosses a fair coin and doubles his stake at every toss. S_t is the gambler's fortune at time t.

If $S_{t+1} = S_t + 2^t$, we say the gambler has **won toss $t+1$**.

If $S_{t+1} = S_t - 2^t$, we say the gambler has **lost toss $t+1$**.

You may assume the following result:

$$\sum_{t=0}^{n-1} 2^t = 2^n - 1.$$

(a) Suppose that the **first** toss that the gambler wins is toss $n+1$. Clearly, this means $S_t \leq x$ for all $t = 0, 1, \ldots, n$. Show that $S_{n+1} = x + 1$, regardless of the value of n. (2)

(b) Show that the process $S_0, S_1, S_2, \ldots$ is a martingale. (3)

The gambler makes the following stopping rules:

- Let L be a lower limit with $L < x$, such that there exists t with $\mathbb{P}(S_t = L) > 0$.
- Stop when the process reaches L ***or*** when the process reaches $x + 1$.

Let T be the time at which the process stops: so $T = t$ if $S_t = L$ or $S_t = x + 1$ and $S_u \notin \{L, x+1\}$ for $u \leq t$. It is clear that T is a stopping time with respect to $\{S_t\}$. You may assume that $\mathbb{E}(T) < \infty$.

Let $M_t = S_{T \wedge t}$ be the stopped process, such that

$$M_t = \begin{cases} S_t & \text{for } t < T, \\ S_T & \text{for } t \geq T. \end{cases}$$

We say that the gambler **wins the game** if he stops in state $x + 1$, and **loses the game** if he stops in state L.

(c) Explain why $L \leq M_t \leq x + 1$ for all $t \leq T$. Hence deduce that there exists a constant c such that $\mathbb{E}(|M_{t+1} - M_t|) \leq c$ for all t. (3)

(d) Part (c) shows that the martingale $\{M_t\}$ satisfies the conditions of the Optional Stopping Theorem. Let w be the probability that the gambler **wins the game** (stops in state $x + 1$). Use the Optional Stopping Theorem to find w. (3)

(e) The gambler now decides that, if he reaches state $x + 1$, he will begin the game again, with the same rules, starting from time $t = 0$ but now with lower limit $L + 1$ and upper limit $x + 2$. If he reaches $x + 2$, he will begin again with lower limit $L + 2$ and upper limit $x + 3$. He will continue in this way until he has finally reached $x + 10$, unless he loses first. Find the gambler's probability of reaching state $x + 10$. Evaluate the probability of finishing with a profit of 10, when the gambler starts with $x = 15$ and sets his initial lower limit as $L = 8$. (3)

Total: 24