Stats 721   Test   Semester 2, 2008

1) Define the Markov chain $X_0, X_1, X_2, \ldots$ such that

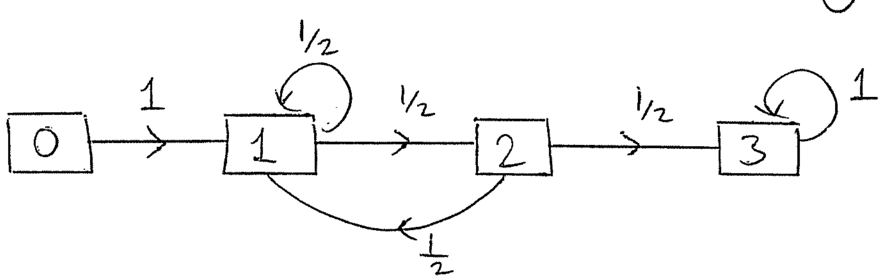
$$X_t = \text{run length at toss } t,$$

where "run length" is defined as the number of consecutive tosses of the same outcome (0 or 1) up to and including toss t.

We define $X_0 = 0, \quad X_1 = 1, \quad X_2 = \begin{cases} 2 & \text{if toss 2} = \text{toss 1}, \\ 1 & \text{if toss 2} \neq \text{toss 1}, \end{cases}$

and so on.

We can stop when $X_t = 3$, because this is the desired end state, so the state space for $\{X_t\}$ can be taken as $\{0, 1, 2, 3\}$ and the transition diagram below:



Define: $p_t = \mathbb{P}(\text{reach state 3 in} \leq t \text{ tosses, starting from state 0})$

Ultimately, we want to find p_{15} and a general formula for p_t.

Define: $a_t = \mathbb{P}(\text{reach state 3 in} \leq t \text{ tosses, starting from state 1})$

$b_t = \mathbb{P}($ " " " " " " " " " $2)$

1 cont) By first-step analysis:

① $p_t = a_{t-1}$

② $a_{t-1} = \frac{1}{2} a_{t-2} + \frac{1}{2} b_{t-2}$

③ $b_{t-2} = \frac{1}{2} a_{t-3} + \frac{1}{2}$ as long as $t-2 \geqslant 1$, i.e. $t \geqslant 3$.

③ in ②: $a_{t-1} = \frac{1}{2} a_{t-2} + \frac{1}{4} a_{t-3} + \frac{1}{4}$

$\Rightarrow 4a_{t-1} = 2a_{t-2} + a_{t-3} + 1$ for $t \geqslant 3$

$\Rightarrow \underline{4a_t = 2a_{t-1} + a_{t-2} + 1}$ $\underline{\text{for } t \geqslant 2}$ ⊛

Clearly, $\underline{a_0 = a_1 = 0}$

Thus we need to solve the equation ⊛:

$$\underline{\underline{4a_t - 2a_{t-1} - a_{t-2} = 1}} \quad ⊛ \quad t \geqslant 2$$

Boundaries $a_0 = a_1 = 0$.

<u>Homogeneous Equation</u>:

Consider $4u_t - 2u_{t-1} - u_{t-2} = 0$ (H)

<u>Characteristic equation</u>:

$$4u^2 - 2u - 1 = 0$$

$$\Rightarrow u = \frac{2 \pm \sqrt{4 + 16}}{8}$$

$$\underline{u = \frac{1 \pm \sqrt{5}}{4}}$$ (Distinct real roots)

i) cont) The two solutions are:

$$u_t = \left(\frac{1+\sqrt{5}}{4}\right)^t \qquad v_t = \left(\frac{1-\sqrt{5}}{4}\right)^t.$$

Check for linear independence:

$$u_0 = v_0 = 1$$

$$u_1 = \frac{1+\sqrt{5}}{4} \qquad v_1 = \frac{1-\sqrt{5}}{4}$$

$$\begin{vmatrix} u_0 & v_0 \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{4} & \frac{1-\sqrt{5}}{4} \end{vmatrix} = \frac{1}{4}\left\{1-\sqrt{5} - 1 - \sqrt{5}\right\} \neq 0.$$

So linear independence holds.

So the general solution of (H) is:

$$u_t = A\left(\frac{1+\sqrt{5}}{4}\right)^t + B\left(\frac{1-\sqrt{5}}{4}\right)^t. \qquad (a)$$

Particular solution to (*):

By inspection, $a_t = 1$ is a particular solution. (b)

General solution of (*):

Adding (a) and (b) gives:

$$a_t = A\left(\frac{1+\sqrt{5}}{4}\right)^t + B\left(\frac{1-\sqrt{5}}{4}\right)^t + 1. \qquad (**)$$

Boundary conditions:

$$a_0 = A + B + 1 = 0 \Rightarrow A + B = -1 \quad (i)$$

$$a_1 = A\left(\frac{1+\sqrt{5}}{4}\right) + B\left(\frac{1-\sqrt{5}}{4}\right) + 1 = 0$$

$$\Rightarrow \quad A(1+\sqrt{5}) + B(1-\sqrt{5}) + 4 = 0$$

$$\Rightarrow \quad (A+B) + \sqrt{5}(A-B) + 4 = 0 \quad \text{(ii)}$$

Using (i) in (ii) $\Rightarrow \quad -1 + \sqrt{5}(A-B) + 4 = 0$

giving : $\quad A - B = -\frac{3}{\sqrt{5}} \quad \text{(iii)}$

also : $\quad A + B = -1 \quad \text{(i)}$

(iii) + (i) $\Rightarrow \quad 2A = -1 - \frac{3}{\sqrt{5}}$

$$A = \frac{-\sqrt{5} - 3}{2\sqrt{5}} = -\left(\frac{5 + 3\sqrt{5}}{10}\right)$$

Similarly, $\quad B = -1 - A$

$$= \frac{5 + 3\sqrt{5} - 10}{10}$$

$$B = \frac{3\sqrt{5} - 5}{10}$$

So the solution of ⊛ is $P_t = a_{t-1}$:

$$P_t = -\left(\frac{3\sqrt{5}+5}{10}\right)\left(\frac{1+\sqrt{5}}{4}\right)^{t-1} + \left(\frac{3\sqrt{5}-5}{10}\right)\left(\frac{1-\sqrt{5}}{4}\right)^{t-1} + 1$$

(general formula required).

Putting $t = 15$ gives:

$$P_{15} = 0.939758$$

2a) Consider $S_t = x + Y_1 + \dots + Y_t$

where $Y_t = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-2p \\ -1 & \text{w.p. } p \end{cases}$

$Y_1, Y_2, \dots$ are independent. Note $\mathbb{E}(Y_t) = 0$.

Check whether $S_0, S_1, S_2, \dots$ is a martingale:

i) Because $|Y_t| \le 1$, then $\underline{\mathbb{E}(|S_t|) \le x + t < \infty} \; \forall t$.

ii) Consider

$$\mathbb{E}(S_{t+1} \mid S_0, \dots, S_t) = \mathbb{E}(S_t + Y_{t+1} \mid S_t)$$

$$= S_t + \mathbb{E}(Y_{t+1}) \quad \begin{pmatrix} Y_{t+1} \text{ indept} \\ \text{of } S_t \end{pmatrix}$$

$$\underline{= S_t.}$$

So $\mathbb{E}(|S_t|) < \infty$ for all t,

and $\mathbb{E}(S_{t+1} \mid S_0, \dots, S_t) = S_t$,

so $\underline{\underline{S_0, S_1, S_2, \dots \text{ is a martingale w.r.t. itself.}}}$

By the optional stopping theorem:

$$\mathbb{E}(S_T) = \mathbb{E}(S_0) = x. \qquad (1)$$

Now $S_T = \begin{cases} 0 & \text{w.p. } 1-w \\ N & \text{w.p. } w \end{cases}$

So (1) $\Rightarrow x = Nw \Rightarrow \underline{\underline{w = \frac{x}{N}}}$ as required.

2b) Need to show:

i) $\mathbb{E}(|M_t|) < \infty$ for all t,

ii) $\mathbb{E}(M_{t+1} \mid S_0, \dots, S_t) = M_t$.

i) $M_t = S_t^2 - 2pt$, so by part (a), $\mathbb{E}(|M_t|) < \infty$ because $\mathbb{E}(|S_t|) < \infty$.

ii) Consider $M_{t+1} = \begin{cases} (S_t+1)^2 - 2p(t+1) & \text{w.p. } p \\ S_t^2 - 2p(t+1) & \text{w.p. } 1-2p \\ (S_t-1)^2 - 2p(t+1) & \text{w.p. } p \end{cases}$

Thus $\mathbb{E}(M_{t+1} \mid S_0, \dots, S_t)$

$$= p\{(S_t+1)^2 + (S_t-1)^2\} + (1-2p)S_t^2 - 2p(t+1)$$

$$= p\{2S_t^2 + 2\} + S_t^2 - 2pS_t^2 - 2pt - 2p$$

$$= S_t^2 - 2pt$$

$$= \underline{\underline{M_t}} \text{ by definition.}$$

So (i) and (ii) are proved, so $M_0, M_1, \dots$ is a martingale w.r.t. $S_0, S_1, \dots$.

2c) Use martingale $M_0, M_1, \dots$ from (b), because it is the only one which involves time, t.

Now OST $\Rightarrow$ $\mathbb{E}(M_T) = \mathbb{E}(M_0) = x^2 - 0 = x^2$. (2)

Further, $S_T^2 = \begin{cases} N^2 \text{ with probability} & w = x/N \\ 0^2 \text{ with probability} & 1-w = 1 - x/N. \end{cases}$

So $\mathbb{E}(S_T^2) = \frac{x}{N} \cdot N^2 = Nx$.

2c cont) So ② gives:

$$x^2 = \mathbb{E}(M_T) = \mathbb{E}(S_T^2 - 2pT)$$

$$x^2 = Nx - 2p\,\mathbb{E}(T)$$

$$\therefore \quad \mathbb{E}(T) = \frac{Nx - x^2}{2p}$$

$$\text{or} \quad \mathbb{E}(T) = \frac{x(N-x)}{2p}$$