1) $m_x = \mathbb{E}(\#\text{steps to finish} \mid \text{start at state } x)$.

Boundaries: $m_0 = 0 \qquad m_N = 0$

Difference equation: by 1st-step analysis,

$$m_x = 1 + p\,m_{x+1} + q\,m_{x-1} + (1-p-q)\,m_x$$

$$\therefore \quad p\,m_{x+1} - (p+q)\,m_x + q\,m_{x-1} = -1 \qquad \text{(E)}$$

Valid for $x = 1, 2, \ldots, N-1$.

Solution of (E):

i) Homogeneous Equation

Consider $p\,u_{x+1} - (p+q)\,u_x + q\,u_{x-1} = 0 \qquad \text{(H)}$

Characteristic Equation:

$$p u^2 - (p+q)u + q = 0 \qquad (*)$$

Note that $u=1$ satisfies this equation:

so we have

$$(pu - q)(u-1) = 0$$

$$\therefore \quad u = 1 \quad \text{and} \quad u = \frac{q}{p}.$$

We know that $p \neq q$, so these are distinct real roots.

1cont.) The two solutions are:

$$\underline{u_x = 1^x = 1} \qquad \underline{v_x = \left(\frac{q}{p}\right)^x}$$

<u>Check for linear independence:</u>

$u_0 = 1^0 = 1 \qquad u_1 = 1^1 = 1$

$v_0 = \left(\frac{q}{p}\right)^0 = 1 \qquad v_1 = \left(\frac{q}{p}\right)^1 = \frac{q}{p}$

So $\begin{vmatrix} u_0 & v_0 \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & q/p \end{vmatrix} = \frac{q}{p} - 1 \neq 0$ because $p \neq q$.

So the general solution of (H) is:

$$\underline{\underline{u_x = A + B\left(\frac{q}{p}\right)^x}}.$$

<u>ii) Particular solution of (E):</u>

Try $m_x = c$ (constant):

$$\Rightarrow \quad pc - (p+q)c + qc = -1$$

$$0c = -1 \quad \text{✗ } \underline{\text{Doesn't work.}}$$

Try $m_x = cx$:

$$\Rightarrow \quad pc(x+1) - (p+q)cx + qc(x-1) = -1$$

$$c\{\cancel{px} + p - \cancel{px} - \cancel{qx} + \cancel{qx} - q\} = -1$$

$$c(p-q) = -1$$

$$\therefore \quad \underline{\underline{c = \frac{1}{q-p}}}$$

OK because $q \neq p$.

1 cont.) So particular solution of (E) is

$$m_x = \frac{x}{q-p}$$

iii) General Solution of (E):

Add Homogeneous Soln + Particular Soln;

$$m_x = A + B\left(\frac{q}{p}\right)^x + \frac{x}{q-p}. \qquad (**)$$

iv) Boundaries:

$$m_0 = 0 \Rightarrow A + B + 0 = 0$$

$$\Rightarrow B = -A \qquad (1)$$

$$m_N = 0 \Rightarrow A + B\left(\frac{q}{p}\right)^N + \frac{N}{q-p} = 0$$

Subst (1) $\Rightarrow$

$$A\left\{1-\left(\frac{q}{p}\right)^N\right\} = \frac{N}{p-q}$$

$$\therefore \quad A = \frac{N}{p-q} \cdot \frac{1}{1-\left(\frac{q}{p}\right)^N}$$

v) Final Solution:

$$m_x = \frac{N}{p-q} \cdot \frac{1-(q/p)^x}{1-(q/p)^N} - \frac{x}{p-q}.$$

vi) $\mathbb{E}(T) = m_x = \frac{N}{p-q}\left\{\frac{1-(q/p)^x}{1-(q/p)^N}\right\} - \frac{x}{p-q}.$

2a) Because $p \neq q$, suspect that

$$M_t = \left(\frac{q}{p}\right)^{S_t} \text{ will be a martingale.}$$

Proof: We know $|S_t| = |x + Y_1 + \dots + Y_t|$

$$\leq x + |Y_1| + \dots + |Y_t|$$

$$\therefore |S_t| \leq x + t$$

so $\underline{\mathbb{E}(|S_t|) < \infty \text{ for all } t.}$

Thus $\underline{\underline{\mathbb{E}(|M_t|) < \infty \text{ also}}}$.

Now consider

$$M_{t+1} = \begin{cases} \left(\frac{q}{p}\right)^{S_t+1} & \text{with probability } p \\ \left(\frac{q}{p}\right)^{S_t} & \text{" \quad " \quad } 1-p-q \\ \left(\frac{q}{p}\right)^{S_t-1} & \text{" \quad " \quad } q \end{cases}$$

So
$$\begin{aligned} \mathbb{E}(M_{t+1} \mid S_t, \dots, S_0) &= p \cdot \left(\frac{q}{p}\right)^{S_t+1} + (1-p-q)\left(\frac{q}{p}\right)^{S_t} + q\left(\frac{q}{p}\right)^{S_t-1} \\ &= \left(\frac{q}{p}\right)^{S_t} \left\{ p \cdot \frac{q}{p} + 1-p-q + q \cdot \frac{p}{q} \right\} \\ &= \left(\frac{q}{p}\right)^{S_t} \left\{ q + 1-p-q+p \right\} \\ &= \left(\frac{q}{p}\right)^{S_t} \end{aligned}$$

$$\therefore \underline{\underline{\mathbb{E}(M_{t+1} \mid S_t, \dots, S_0) = M_t}}$$

So $\underline{\underline{M_0, M_1, \dots \text{ is a martingale with respect to } S_0, S_1, \dots}}$

2b) By the optional stopping theorem, and because $T = \{\text{time of first reaching state } 0 \text{ or } N\}$ is a stopping time with respect to $S_0, S_1, \ldots,$

we have:

$$\mathbb{E}(M_T) = \mathbb{E}(M_0)$$

$$\therefore \quad \underline{\mathbb{E}(M_T) = \left(\frac{q}{p}\right)^x} \quad \text{because } M_0 = \left(\frac{q}{p}\right)^x. \quad \circledast$$

But

$$M_T = \begin{cases} \left(\frac{q}{p}\right)^0 & \text{with probability } 1-w \\ \left(\frac{q}{p}\right)^N & \text{" " } w \end{cases}$$

So $\circledast \Rightarrow$

$$(1-w)\left(\frac{q}{p}\right)^0 + w\left(\frac{q}{p}\right)^N = \left(\frac{q}{p}\right)^x$$

$$\Rightarrow \quad 1 - \left(\frac{q}{p}\right)^x = w\left\{1 - \left(\frac{q}{p}\right)^N\right\}$$

$$\therefore \quad \underline{\underline{w = \frac{1-(q/p)^x}{1-(q/p)^N}}}$$

as stated.

2c) Now

$$S_T = x + \sum_{i=1}^{T} Y_i$$

so

$$\mathbb{E}(S_T) = x + \mathbb{E}(Y_i)\,\mathbb{E}(T) \quad \text{using Wald's equation.} \quad \circledast$$

Now

$$S_T = \begin{cases} 0 & \text{w.p. } 1-w \\ N & \text{w.p. } w \end{cases}$$

so

$$\underline{\mathbb{E}(S_T) = Nw = \frac{N\left(1-\left(\frac{q}{p}\right)^x\right)}{1-\left(\frac{q}{p}\right)^N}} \quad \text{using (b).}$$

2c (cont) Note that $S_0, S_1, \ldots$ need not be a martingale for this to apply (and it isn't).

Also, $Y_i = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p-q \\ -1 & \text{w.p. } q \end{cases}$

so $\underline{\mathbb{E}(Y_i) = p - q}$.

Thus by ⊛,

$$\mathbb{E}(S_T) = \frac{N\left\{1-\left(\frac{q}{p}\right)^x\right\}}{1-(q/p)^N} = x + (p-q)\,\mathbb{E}(T)$$

$$\Rightarrow \quad \mathbb{E}(T) = \frac{N}{p-q}\left\{\frac{1-(q/p)^x}{1-(q/p)^N}\right\} - \frac{x}{p-q}$$

This is the same answer as in Q1, and it should be the same.