STATS 721 — Test Solutions — Semester 2, 2010

1) First-step analysis:

$$m_x = 1 + \tfrac{2}{3} m_{x+1} + \tfrac{1}{3} m_{x-1}$$

$$\therefore \quad 2m_{x+1} - 3m_x + m_{x-1} = -3 \qquad (E) \text{ to solve.}$$

Boundaries: $m_0 = 0, \; m_N = 0.$

Homogeneous Equation:

$$2u_{x+1} - 3u_x + u_{x-1} = 0 \qquad (H)$$

Characteristic Equation:

$$2u^2 - 3u + 1 = 0$$

$$(2u - 1)(u - 1) = 0$$

$$\therefore \quad u = \tfrac{1}{2} \quad \text{or} \quad u = 1.$$

Two solutions: $u_x = \left(\frac{1}{2}\right)^x$ and $v_x = 1^x = 1.$

Check for linear independence:

$$\begin{vmatrix} u_0 & v_0 \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{vmatrix} = 1 - \tfrac{1}{2} \neq 0.$$

So the general solution of (H) is:

$$u_x = A\left(\tfrac{1}{2}\right)^x + B.$$

Particular solution to (E):

Try $m_x = c$ (constant) $\Rightarrow$ require $2c - 3c + c = -3$

$\Rightarrow \quad 0 = -3$, not possible.

1 cont.) Try $m_x = cx$: require c such that

$$2m_{x+1} - 3m_x + m_{x-1} = -3$$
$$\Rightarrow 2c(x+1) - 3cx + c(x-1) = -3$$
$$c\{2x + x - 3x + 2 - 1\} = -3$$
$$\therefore \quad c = -3$$

So a particular solution to E is:

$$m_x = -3x.$$

General solution of E:

General solution = (Homogeneous general soln) + (Particular Soln)

$$\Rightarrow \quad m_x = A\left(\tfrac{1}{2}\right)^x + B - 3x.$$

Boundary conditions:

$$m_0 = 0 \Rightarrow A + B = 0 \quad \text{So} \quad B = -A.$$
$$m_N = 0 \Rightarrow A\left(\tfrac{1}{2}\right)^N + B - 3N = 0$$
$$\Rightarrow A\left\{\frac{1}{2^N} - 1\right\} = 3N$$
$$\Rightarrow \quad A = \frac{3N}{\frac{1}{2^N} - 1}$$

Final solution for m_x :

$$m_x = \left\{\frac{3N}{\frac{1}{2^N} - 1}\right\}\left(\frac{1}{2^x} - 1\right) - 3x \qquad x = 0, 1, \ldots, N.$$

2) a) If tosses $1, 2, \ldots, n$ are lost, and toss $n+1$ is won, then

$$S_{n+1} = \underset{\text{toss}}{x} - \underset{1}{2^0} - \underset{2}{2^1} - \underset{\cdots}{\cdots} - \underset{n}{2^{n-1}} + \underset{n+1}{2^n}$$

So $$S_{n+1} = x - \sum_{t=0}^{n-1} 2^t + 2^n$$

$$= x - \{2^n - 1\} + 2^n \quad \text{using Hint}$$

$\underline{\underline{S_{n+1} = x+1}}$ as stated.

b) We require $\mathbb{E}(|S_n|) < \infty$ for all n, ①

and $\mathbb{E}(S_{t+1} \mid S_0, \ldots, S_t) = S_t$ for all t. ②

Proof of ①

$$|S_n| \leqslant x + 2^0 + \cdots + 2^{n-1}$$

$$= x + \sum_{t=0}^{n-1} 2^t$$

$\therefore |S_n| \leqslant x + 2^n - 1 < \infty$ for all n.

So $\underline{\underline{\mathbb{E}(|S_n|) < \infty \text{ for all } n.}}$

Proof of ②

Consider $$S_{t+1} = \begin{cases} S_t + 2^t & \text{w.p. } \frac{1}{2} \\ S_t - 2^t & \text{w.p. } \frac{1}{2} \end{cases}$$

So $$\mathbb{E}(S_{t+1} \mid S_0, \ldots, S_t) = \tfrac{1}{2}\{S_t + 2^t\} + \tfrac{1}{2}\{S_t - 2^t\}$$

$\therefore \underline{\underline{\mathbb{E}(S_{t+1} \mid S_0, \ldots, S_t) = S_t}}$ as required.

So $\underline{\underline{S_0, S_1, S_2, \ldots \text{ is a martingale.}}}$

2c) $S_0, S_1, S_2, \ldots$ is monotonically decreasing until the first win occurs.

$\{M_t\}$ stops either at L (which must occur before the first win) or at $x+1$ (which must occur on the first win).

Thus if $\{M_t\}$ stops at L, we have

$$x = M_0 > M_1 > \cdots > M_T = L$$

and if $\{M_t\}$ stops at $x+1$, we have

$$M_T = x+1 > M_0 > M_1 > \cdots > \underbrace{M_{T-1} > L}$$

because the chain does not reach L before time T.

Overall, $\underline{\underline{L \leq M_t \leq x+1 \text{ for all } t \leq T.}}$

So $|M_{t+1} - M_t| \leq x+1-L$ necessarily, for all $t \leq T$.

So $\underline{\underline{\mathbb{E}(|M_{t+1} - M_t|) \leq x+1-L \text{ for all } t,}}$

noting that $M_{t+1} - M_t = 0$ for $t \geq T$.

d) By the Optional Stopping Theorem,

$$\mathbb{E}(M_T) = \mathbb{E}(M_0) = x \text{ in this case.} \quad (*)$$

2d cont.) Now

$$M_T = \begin{cases} x+1 & \text{with prob. } w \quad \text{(win)} \\ L & \text{" \quad " } 1-w \quad \text{(lose)} \end{cases}$$

So $\quad \mathbb{E}(M_T) = w(x+1) + (1-w)L$

$$\mathbb{E}(M_T) = w(x+1-L) + L.$$

In ⊛ $\Rightarrow \quad w(x+1-L) + L = x$

$$\therefore \quad \mathbb{P}(\text{win}) = w = \frac{x-L}{x+1-L}$$

e) Starting from $x+k$ with lower limit $L+k$, using (d) gives

$$\mathbb{P}(\text{win}) = \frac{x+k-(L+k)}{x+k+1-(L+k)} = \frac{x-L}{x+1-L} = w.$$

So each time the gambler starts, he has probability w of winning.

To win $x+10$ we need 10 wins consecutively. All games are independent.

So $\quad \mathbb{P}(\text{reaches } x+10) = w^{10} = \left(\frac{x-L}{x+1-L}\right)^{10}.$

When $x=15$, $L=8$,

$$\mathbb{P}(\text{reaches } 25) = \left(\frac{15-8}{16-8}\right)^{10} = (0.875)^{10} = 0.263.$$