# THE UNIVERSITY OF AUCKLAND

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**SECOND SEMESTER, 2003**
**Campus: City**

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**STATISTICS**

**Stochastic Processes**
**Topics in Statistics 2**

**(Time allowed: THREE hours)**

**NOTE:** Attempt **ALL** questions. Marks for each question are shown in brackets.
An Attachment containing useful information is found on page 5.

1. (a) Let $A$ and $B$ be independent events. Show that $A$ and $\overline{B}$ are also independent events. (2)

   (b) Let $A$ and $B$ be any events, with $\mathbb{P}(A) = \frac{4}{5}$ and $\mathbb{P}(B) = \frac{1}{2}$. Show that

$$\frac{3}{10} \leq \mathbb{P}(A \cap B) \leq \frac{1}{2}.$$

(4)

2. A mouse is in a room with four exits at the centre of a maze.

   - Exit 1 leads outside the maze after 4 minutes.
   - Exit 2 leads back to the room after 2 minutes.
   - Exit 3 leads back to the room after 5 minutes.
   - Exit 4 leads back to the room after 6 minutes.

   Every time the mouse makes a choice, it is equally likely to choose any of the four exits. Let $T$ be the total time taken for the mouse to leave the maze.

   (a) Find $\mathbb{E}(T)$. (3)

   (b) Given that the mouse last left the room 3 minutes ago, find the probability that the mouse has chosen the escape route (Exit 1). (2)

   (c) Starting at the room, find the probability that the mouse escapes in exactly 6 minutes. (2)

3. The owner of a cafe is trying to decide how much to charge for a breakfast deal. He has to charge at least \$10 to cover costs. He wishes to charge $\$(10+x)$ so that he makes $\$x$ profit. He needs to make $x$ large enough to make a reasonable profit, but small enough to encourage customers to return after each visit.

   The owner believes that

$$\mathbb{P}(\text{customer returns for another visit}) = \begin{cases} \dfrac{4}{5} - \dfrac{x^2}{125} & \text{if } 0 \le x \le 10, \\ 0 & \text{otherwise.} \end{cases}$$

   The customer returns after each visit according to the probability above. If the customer fails to return after a visit, they will never return again.

   Let $P$ be the total profit that the owner makes from all visits of a given customer, starting at the customer's first visit and ending at the point that the customer fails to return.

   (a) Show that $\mathbb{E}(P) = \dfrac{125x}{25+x^2}$. (4)

   (b) By differentiating the expression above, find the value of $x$ that the owner should choose. (You may assume without verification that any stationary points in the range $0 \le x \le 10$ are maxima.) (3)

4. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3\}$, with transition matrix

$$P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

   (a) Draw the transition diagram. (2)

   (b) Find an equilibrium distribution for $P$. (3)

   (c) Does $X_t$ converge to the distribution in (b) as $t \to \infty$? Explain why or why not. (2)

5. (a) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common probability generating function $G_X(s)$. Let $N$ be a random variable, independent of the $X_i$'s, with PGF $G_N(s)$, and let $T = X_1 + \ldots + X_N$ (the sum of a random number of random variables). Show that the PGF of $T$ is:

$$G_T(s) = G_N\Big(G_X(s)\Big).$$

   (5)

   (b) Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where $Z_n$ denotes the number of individuals born at time $n$, and $Z_0 = 1$. Let $Y$ be the family size distribution, with PGF $G(s)$. Let $G_n(s)$ be the probability generating function of $Z_n$. Show that

$$G_n(s) = G_{n-1}\Big(G(s)\Big).$$

   (2)

   (c) In the branching process defined in part (b), let $p_i = \mathbb{P}(Y = i)$ for $i = 0, 1, 2, \ldots$ Show that

$$\mathbb{P}(Z_n = 1) = p_1 G'_{n-1}(p_0),$$

   and find a similar expression for $\mathbb{P}(Z_n = 2)$. (8)

6. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where $Z_n$ denotes the population size at time $n$, and $Z_0 = 1$. Let $Y$ be the family size distribution. Suppose that $Y \sim \text{Geometric}(p = 0.5)$, so that

$$\mathbb{P}(Y = y) = \left(\frac{1}{2}\right)^{y+1} \qquad \text{for } y = 0, 1, 2, \ldots$$

(a) Let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of $Y$. Show that

$$G(s) = \frac{1}{2 - s},$$

and state the radius of convergence. (3)

(b) Find the probability of eventual extinction, $\gamma$. (3)

(c) Let $G_n(s) = \mathbb{E}\left(s^{Z_n}\right)$ be the PGF of the population size at time $n$. It can be shown that

$$G_n(s) = \frac{n - (n-1)s}{(n+1) - ns}.$$

Additionally, it can be shown that

$$\mathbb{P}(Z_n = r) = \begin{cases} \dfrac{n}{n+1} & \text{for } r = 0, \\ \dfrac{n^{r-1}}{(n+1)^{r+1}} & \text{for } r = 1, 2, 3, \ldots. \end{cases}$$

Using the expression for $G_n(s)$, verify that the expression for $\mathbb{P}(Z_n = r)$ is correct for $r = 0$ and $r = 1$. (4)

(d) Find the probability that the population becomes extinct *at* generation 8. (3)

(e) Using the expressions given above, and relevant information from the Attachment, find $\mathbb{P}(Z_{10} \leq 20)$. (5)

7. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3, 4, 5, 6\}$, with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(a) Draw the transition diagram, and identify all communicating classes. For each class, state whether or not it is closed. (5)

(b) Let the set $A = \{1\}$. Find the vector of hitting probabilities, $\boldsymbol{h_A} = (h_{1A}, \ldots, h_{6A})^T$, where $h_{iA}$ is the probability of eventually hitting the set $A$, starting from state $i$. (6)

8. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}, \quad \text{where } 0 < \alpha < 1, \;\; 0 < \beta < 1.$$

The general solution for $P^t$ is as follows:

$$P^t = \frac{1}{\alpha+\beta}\left\{ \begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} + \begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix} (1-\alpha-\beta)^t \right\}.$$

(a) Suppose that $X_0 \sim (0.1, 0.9)^T$. Find a vector describing the distribution of $X_1$. (2)

(b) Find $\mathbb{P}(X_1 = 2, X_2 = 1, X_3 = 2 \,|\, X_0 = 2)$. (2)

(c) Let $\{Y_0, Y_1, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3\}$, with transition matrix

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

By reformulating the three-state chain as a suitable two-state chain, and using the general solution for $P^t$ given above, find a general formula for $\mathbb{P}(Y_t = 1 \,|\, Y_0 = 1)$ for any $t$. (8)

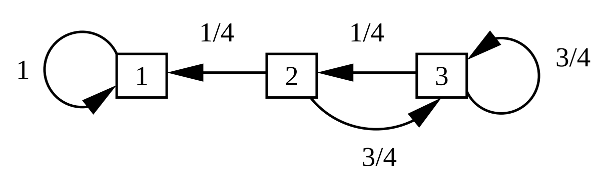
9. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a process that behaves like a branching process, except for one important change: the distribution of family size is not constant for all generations, but depends upon the value of $Z_t$. Specifically, if there are $Z_t > 0$ individuals in generation $t$, then each of these $Z_t$ individuals has the following family size distribution:

$$Y^{(t)} = \begin{cases} 0 & \text{with probability } \left(1 - \frac{1}{Z_t}\right), \\ 1 & \text{with probability } \frac{1}{Z_t}. \end{cases}$$

If $Z_t = 0$, then $Z_{t+1} = 0$ with probability 1.

Suppose that $Z_0 = 3$, i.e. the process starts with 3 individuals at time 0. Considering $\{Z_0, Z_1, Z_2, \ldots\}$ as a Markov chain on the state space $S = \{0, 1, 2, 3\}$, find the transition matrix, and hence find the probability of eventual extinction starting from $Z_0 = 3$. (12)

10. Let $\{X_0, X_1, \ldots\}$ be a Markov chain with transition diagram below.




Define the random variable $T_k$ to be the number of steps taken to hit state 1, starting from state $k$, for $k = 1, 2, 3$. Let $G_k(s) = \mathbb{E}(s^{T_k})$ be the probability generating function of $T_k$. By conditioning on the outcome of the first step, and using ideas of conditional expectation, find $G_3(s)$. (5)

# ATTACHMENT

1. **Discrete Probability Distributions**

| **Distribution** | $\mathbb{P}(X = x)$ | $\mathbb{E}(X)$ | PGF, $\mathbb{E}(s^X)$ |
|---|---|---|---|
| Geometric($p$) | $pq^x$ (where $q = 1 - p$), for $x = 0, 1, 2, \ldots$ | $\dfrac{q}{p}$ | $\dfrac{p}{1 - qs}$ |
| | Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$. | | |
| Binomial($n, p$) | $\binom{n}{x} p^x q^{n-x}$ (where $q = 1 - p$), for $x = 0, 1, 2, \ldots, n$. | $np$ | $(ps + q)^n$ |
| | Number of successes in $n$ independent trials, each with $\mathbb{P}(\text{success}) = p$. | | |
| Poisson($\lambda$) | $\dfrac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \ldots$ | $\lambda$ | $e^{\lambda(s-1)}$ |

2. **Uniform Distribution:** $X \sim \text{Uniform}(a, b)$.
Probability density function, $f_X(x) = \frac{1}{b-a}$ for $a < x < b$. Mean, $\mathbb{E}(X) = \frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

| | | |
|---|---|---|
| **Definition:** | $G_X(s) = \mathbb{E}(s^X)$ | |
| **Moments:** | $\mathbb{E}(X) = G'_X(1)$ | $\mathbb{E}\Big\{X(X-1)\ldots(X-k+1)\Big\} = G_X^{(k)}(1)$ |
| **Probabilities:** | $\mathbb{P}(X = n) = \dfrac{1}{n!} G_X^{(n)}(0)$ | |

4. **Geometric Series:** $1 + r + r^2 + r^3 + \ldots = \sum_{x=0}^{\infty} r^x = \dfrac{1}{1-r}$ for $|r| < 1$.

Finite sum: $\sum_{x=0}^{n} r^x = \dfrac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p, q \in \mathbb{R}$, and integer $n > 0$, $(p + q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\sum_{x=0}^{\infty} \dfrac{\lambda^x}{x!} = e^{\lambda}$.