THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2004
Campus: City

STATISTICS

Stochastic Processes
Topics in Statistics 2

(Time allowed: THREE hours)

NOTE: Attempt **ALL** questions. Marks for each question are shown in brackets.
An Attachment containing useful information is found on page 6.

1. Let A and B be any events.

   (a) Show that $\mathbb{P}(A \cup B) = 1 - \mathbb{P}(\overline{A}) + \mathbb{P}(\overline{A} \cap B)$. (4)

   (b) Starting from the result in part (a), show that $\mathbb{P}(A \cup B) = 1 - \mathbb{P}(\overline{A} \cap \overline{B})$. (4)

2. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3\}$, with transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}.$$

   (a) Draw the transition diagram. (2)

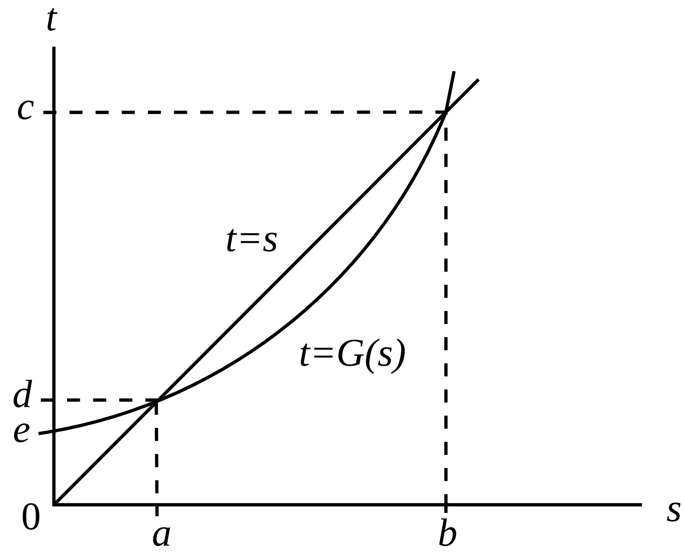
   (b) Find an equilibrium distribution for P. (4)

   (c) Does X_t converge to the distribution in (b) as $t \to \infty$? Explain why or why not. (3)

3. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. Let $\mu = \mathbb{E}(Y)$, let $\sigma^2 = \text{Var}(Y)$, and let γ be the probability of eventual extinction.

The diagram below shows a graph of $G(s)$, with missing values a, b, c, d, and e. Each of the missing values is included on the following list:

$0 \quad 1 \quad \mu \quad \sigma^2 \quad \gamma \quad y \quad Y \quad \mathbb{P}(Y=0) \quad \mathbb{P}(Y=1) \quad \mathbb{P}(Y=y) \quad \mathbb{E}(s^Y)$



(a) By reference to the list above, state the values of a, b, c, d, and e. (5)

(b) Using the diagram above, state whether $\mu > 1$, $\mu = 1$, or $\mu < 1$. Give a reason for your answer. (2)

4. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

The general solution for P^t is as follows:

$$P^t = \begin{pmatrix} \frac{5}{13} & \frac{8}{13} \\ \frac{5}{13} & \frac{8}{13} \end{pmatrix} + \begin{pmatrix} \frac{8}{13} & -\frac{8}{13} \\ -\frac{5}{13} & \frac{5}{13} \end{pmatrix} \left(\frac{7}{20}\right)^t.$$

(a) Suppose that $X_0 \sim (0.4, 0.6)^T$. Find a vector describing the distribution of X_1. (2)

(b) Again suppose that $X_0 \sim (0.4, 0.6)^T$. Find $\mathbb{P}(X_0 = 2,\ X_2 = 2,\ X_4 = 1)$. (4)

(c) Using *only* the formula given for P^t, state whether or not the Markov chain converges to an equilibrium distribution as $t \to \infty$. If the chain does converge to equilibrium, state what the equilibrium distribution is. (3)

5. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and let γ be the probability of ultimate extinction..

(a) Suppose that $Y \sim \text{Geometric}(p = \frac{1}{4})$, so that

$$\mathbb{P}(Y = y) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^y \qquad \text{for } y = 0, 1, 2, \ldots$$

Let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. Show that

$$G(s) = \frac{1}{4 - 3s} \qquad \text{for } |\, s \,| < \frac{4}{3}.$$

(3)

(b) Let $G_2(s)$ be the probability generating function of Z_2. Find $G_2(s)$, and simplify your expression as far as possible. (3)

(c) Find the probability of eventual extinction, γ. (3)

(d) Find the probability that the branching process goes extinct *at* generation $n = 3$. (3)

(e) Now suppose that $Y \sim \text{Poisson}(\lambda = 0.5)$, so that

$$\mathbb{P}(Y = y) = \frac{(0.5)^y}{y!}\, e^{-0.5} \qquad \text{for } y = 0, 1, 2, \ldots$$

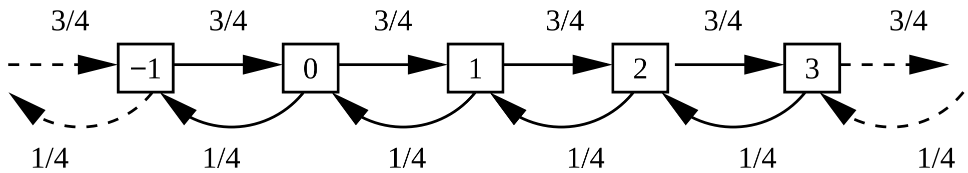
Let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. Show that

$$G(s) = e^{0.5(s-1)} \qquad \text{for } s \in \mathbb{R}.$$

(3)

(f) Using $Y \sim \text{Poisson}(\lambda = 0.5)$, state the probability of eventual extinction, γ. (2)

6. Let $\{X_0, X_1, X_2, \ldots\}$ be a random walk on the integers, with transition diagram below.



Let T be the number of steps taken to reach state 1, starting at state 0. Let $H(s) = \mathbb{E}(s^T)$ be the probability generating function of T.

(a) Show that $H(s)$ must be either the (+) root or the (−) root of the following expression:

$$H(s) = \frac{4 \pm \sqrt{16 - 12s^2}}{2s}.$$

(5)

(b) Simplify the above expression to show that $H(s) = \dfrac{2 \pm \sqrt{4 - 3s^2}}{s}$. (1)

(c) By considering $H(0)$, prove that $H(s)$ can **not** be the (+) root in the expression above. (2)

(d) Using the expression $H(s) = \dfrac{2 - \sqrt{4 - 3s^2}}{s}$, find whether T is a defective random variable. (2)

(e) Using the PGF $H(s)$ as above, find $E(T)$, the expected number of steps taken to reach state 1, starting at state 0. (5)

(f) Let W be the number of steps required to reach state 4, starting from state 0. Define $G(s) = \mathbb{E}(s^W)$ to be the probability generating function of W. Find $G(s)$ in terms of $H(s)$. (2)

7. A fair 6-sided die is tossed t times. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain where X_t represents the *number of distinct values* that have appeared up to and including toss t. The chain starts with $X_0 = 0$ (no distinct values at toss 0). At toss 1, we always have $X_1 = 1$ because one distinct value has always appeared after one toss. To illustrate further, if the sequence of tosses 1 to 5 is:

5 4 4 2 5,

then the Markov chain $\{X_0, X_1, X_2, \ldots\}$ begins:

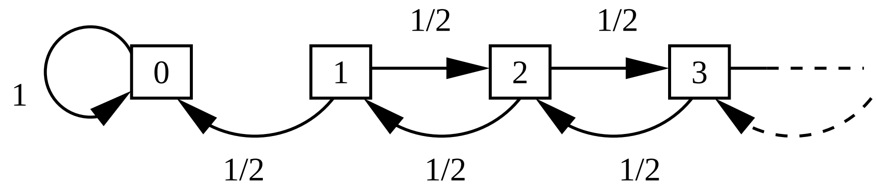
$$X_0 = 0 \quad X_1 = 1 \quad X_2 = 2 \quad X_3 = 2 \quad X_4 = 3 \quad X_5 = 3 \quad \ldots$$

The Markov chain $\{X_0, X_1, X_2, \ldots\}$ has state space $\{0, 1, 2, 3, 4, 5, 6\}$.

(a) Construct the transition diagram of the Markov chain. (4)

(b) Let T be the number of tosses required before *all* values 1, 2, ..., 6 have appeared on the die at least once. Find $\mathbb{E}(T)$. (6)

8. Let $\{X_0, X_1, \ldots\}$ be a Markov chain on $\{0, 1, \ldots\}$ with the following transition diagram:



Given a starting state k, define h_k to be the probability that the process eventually reaches state 0, starting from state k. We aim to prove by mathematical induction that

$$h_k = kh_1 - (k-1) \qquad \text{for } k = 0, 1, 2, \ldots, \qquad (\star)$$

where h_1 is the probability that the process eventually reaches state 0, starting from state 1.

(a) Show that formula $(\star)$ is true for $k = 0$ and $k = 1$. (2)

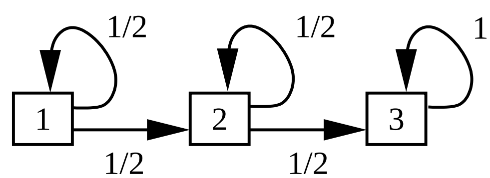
(b) By considering the hitting probability equation for h_r, show that

$$h_{r+1} = 2h_r - h_{r-1} \qquad \text{for } r = 1, 2, 3, \ldots$$ (2)

(c) Suppose that formula $(\star)$ is true for all $k = 0, 1, \ldots, r$, for some integer $r \geq 1$. Using part (b), show that formula $(\star)$ is true for $k = r + 1$. Hence deduce that formula $(\star)$ is true for all $k = 0, 1, 2, \ldots$. (4)

(d) It remains to find the value of h_1. Using the fact that $0 \leq h_k \leq 1$ for all k, find h_1. (3)

9. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $\{1, 2, 3\}$ with transition diagram shown below.



Define random variables T and U as follows.

$$\begin{aligned} T &= \text{number of steps taken to hit state 3, starting from state 1.} \\ U &= \text{number of steps taken to hit state 3, starting from state 2.} \end{aligned}$$

(a) Show that the mean hitting times for state 3 are $\mathbb{E}(U) = 2$ and $\mathbb{E}(T) = 4$. (4)

(b) Find $\mathbb{E}(U^2)$, and hence, or otherwise, find $\mathbb{E}(T^2)$ and $\text{Var}(T)$. (8)

ATTACHMENT

1. **Discrete Probability Distributions**

Distribution	$\mathbb{P}(X = x)$	$\mathbb{E}(X)$	PGF, $\mathbb{E}(s^X)$
Geometric(p)	pq^x (where $q = 1 - p$), for $x = 0, 1, 2, \ldots$	$\frac{q}{p}$	$\frac{p}{1 - qs}$
	Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Binomial(n, p)	$\binom{n}{x} p^x q^{n-x}$ (where $q = 1 - p$), for $x = 0, 1, 2, \ldots, n$.	np	$(ps + q)^n$
	Number of successes in n independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Poisson(λ)	$\frac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \ldots$	λ	$e^{\lambda(s-1)}$

2. **Uniform Distribution:** $X \sim \text{Uniform}(a, b)$.
Probability density function, $f_X(x) = \frac{1}{b-a}$ for $a < x < b$. Mean, $\mathbb{E}(X) = \frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

Definition:	$G_X(s) = \mathbb{E}(s^X)$	
Moments:	$\mathbb{E}(X) = G'_X(1)$	$\mathbb{E}\Big\{ X(X-1)\ldots(X-k+1) \Big\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \frac{1}{n!} G_X^{(n)}(0)$	

4. **Geometric Series:** $1 + r + r^2 + r^3 + \ldots = \sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$ for $|r| < 1$.

Finite sum: $\sum_{x=0}^{n} r^x = \frac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p, q \in \mathbb{R}$, and integer $n > 0$, $(p + q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$.