THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2005
Campus: City

STATISTICS

Stochastic Processes
Topics in Statistics 2

(Time allowed: THREE hours)

NOTE: Attempt **ALL** questions. Marks for each question are shown in brackets. An **Attachment** containing useful information is found on page 8.

1. Let A and B be any events.

(a) Show that $\mathbb{P}(A \cup B) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \overline{B}) + \mathbb{P}(\overline{A} \cap B)$. (4M)

(b) By rearranging the expression in part (a), we obtain:

$$\mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) = \mathbb{P}(A \cap \overline{B}) + \mathbb{P}(\overline{A} \cap B).$$

Give a sentence in plain English to explain what probability the left-hand side and the right-hand side both represent. *Example: if the probability were $\mathbb{P}(A \cap B)$, a suitable sentence would be 'Probability that A and B both occur.'* (2M)

(c) Let A and B be any events with $\mathbb{P}(A) = \frac{1}{3}$ and $\mathbb{P}(B) = \frac{3}{4}$. Show that

$$\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}.$$

[Hint: consider $\mathbb{P}(A \cup B)$.] (4M)

2. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3\}$, with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) Draw the transition diagram. (2E)

(b) Find an equilibrium distribution for P. (4E)

(c) Does X_t converge to the distribution in (b) as $t \to \infty$? Explain why or why not. (2M)

3. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and let γ be the probability of ultimate extinction.

Suppose that $Y \sim \text{Binomial}(n, p)$, so that

$$\mathbb{P}(Y = y) = \binom{n}{y} p^y q^{n-y} \qquad \text{for } y = 0, 1, \ldots, n,$$

where $q = 1 - p$.

(a) Let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. Show that

$$G(s) = (ps + q)^n \qquad \text{for } s \in \mathbb{R}.$$

(3E)

(b) Suppose that $Y \sim \text{Binomial}(n = 2, p = 0.6)$. Let $G_2(s)$ be the probability generating function of Z_2. Write down an expression for $G_2(s)$. (You do *not* need to simplify your answer.) (2E)

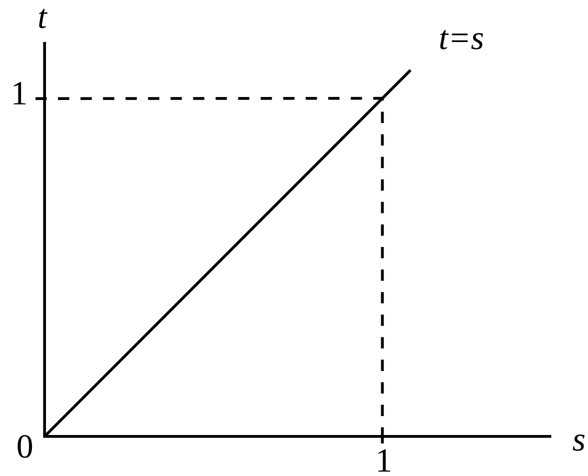
(c) Continue to assume that $Y \sim \text{Binomial}(2, 0.6)$. Find the probability of eventual extinction, γ. (3E)

(d) Find the probability that the branching process goes extinct *by* generation $n = 4$.
[Hint: use part (b). You do *not* need to calculate a general expression for $G_4(s)$.] (4M)

(e) Suppose there are 10 individuals alive at generation 8. What is the probability of eventual extinction? (2M)

4. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. Let $\mu = \mathbb{E}(Y)$, and let γ be the probability of eventual extinction.

The diagram below shows a graph of $t = s$ for $0 \leq s \leq 1$.



(a) Suppose that $\gamma < 1$. Copy the diagram above and mark on it the following features:
   (i) The curve $t = G(s)$;
   (ii) γ;
   (iii) $\mathbb{P}(Y = 0)$;
   (iv) μ. (4E)

(b) State whether $\mu > 1$, $\mu = 1$, or $\mu < 1$. Give a reason for your answer. (2E)

5. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3\}$, with transition matrix P. The general solution for P^t is as follows:

$$P^t = \frac{1}{8}\left\{ \begin{pmatrix} 1 & 3 & 4 \\ 1 & 3 & 4 \\ 1 & 3 & 4 \end{pmatrix} + (-1)^t \begin{pmatrix} 1 & 3 & -4 \\ 1 & 3 & -4 \\ -1 & -3 & 4 \end{pmatrix} \right\} \qquad \text{for } t = 1, 2, 3, \ldots.$$

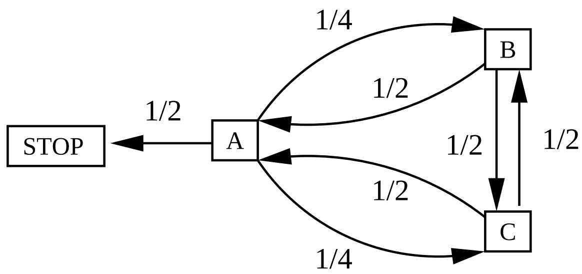
(a) Draw the transition diagram of the Markov chain. (2E)

(b) Suppose that $X_0 \sim \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$. Find a vector describing the distribution of X_1. (2E)

(c) Again suppose that $X_0 \sim \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$. Find $\mathbb{P}(X_0 = 3,\ X_2 = 3,\ X_5 = 1)$. (4M)

(d) Using *only* the formula given for P^t, state whether or not the Markov chain converges to an equilibrium distribution as $t \to \infty$. If the chain does converge to equilibrium, state what the equilibrium distribution is. If the chain does not converge to equilibrium, explain why. (3M)

6. Consider a stochastic process with the following transition diagram.

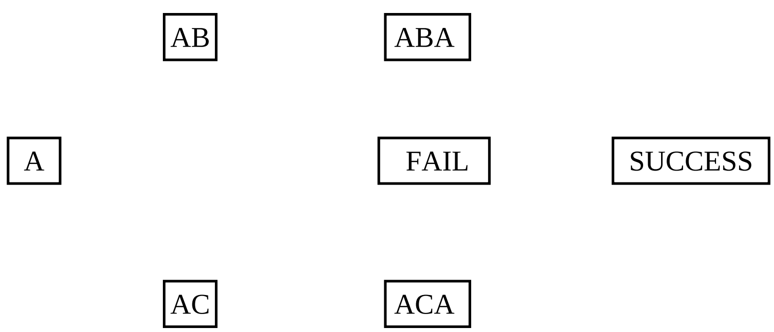


Starting from state A, we wish to find the probability that all states A, B, and C are visited before stopping. We will call this probability p_A.

We aim to find p_A by defining a new stochastic process with the following states:

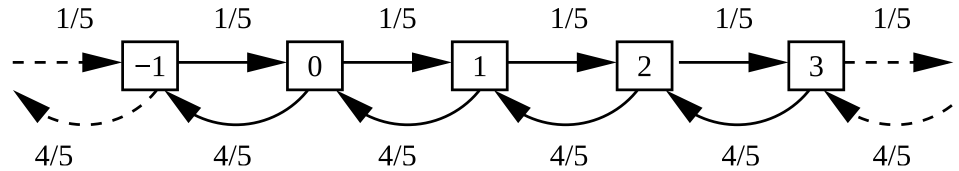
A	:	visited state A only, and currently in state A.
AB	:	visited states A and B, and currently in state B.
ABA	:	visited states A and B, and currently in state A.
AC	:	visited states A and C, and currently in state C.
ACA	:	visited states A and C, and currently in state A.
Success	:	visited all three states A, B, and C.
Fail	:	reached state *Stop* before all three states A, B, and C are visited.

The incomplete transition diagram for the redefined process is shown below.



Copy the incomplete transition diagram above, and add all transition arrows and probabilities to the diagram. Hence find p_A, the probability that all states A, B, and C are visited before stopping, starting from state A. (8M)

7. Let $\{X_0, X_1, X_2, \ldots\}$ be a random walk on the integers, with transition diagram below.



Let T be the number of steps taken to reach state 1, starting at state 0. Let $H(s) = \mathbb{E}(s^T)$ be the probability generating function of T.

(a) Show that $H(s)$ must be either the (+) root or the (−) root of the following expression:

$$H(s) = \frac{5 \pm \sqrt{25 - 16s^2}}{8s}.$$

(5M)

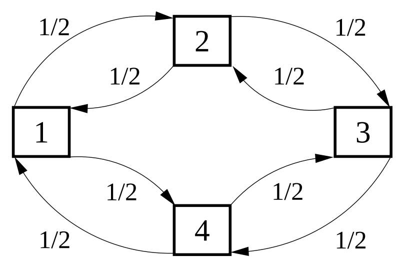
(b) By considering $H(0)$, prove that $H(s)$ can **not** be the (+) root in the expression above. (2M)

(c) Using the expression $H(s) = \dfrac{5 - \sqrt{25 - 16s^2}}{8s}$, find whether T is a defective random variable. (2M)

(d) What is the probability that we *never* reach state 1, starting from state 0? (2M)

(e) Let W be the number of steps required to reach state 3, starting from state 0. Define $G(s) = \mathbb{E}(s^W)$ to be the probability generating function of W. Find $G(s)$ in terms of $H(s)$. (2M)

8. Four friends stand in a circle in the park and throw a frisbee to each other according to the following transition diagram. For example, when person 1 has the frisbee, he throws it to person 2 or to person 4 with probability 1/2 each.



Let T_{ij} be the number of times the frisbee is thrown before it first returns to person j, starting with person i. For example, if the frisbee is thrown person 1 → person 2 → person 1, then $T_{11} = 2$. If the frisbee is thrown person 2 → person 3, then $T_{23} = 1$.

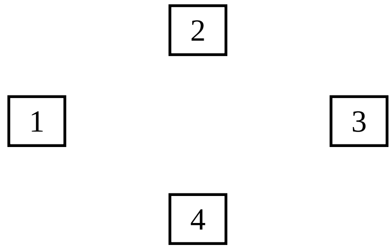
(a) For each of the following pairs of random variables, state whether or not the two random variables have the same distribution as each other.

(i) T_{11} and T_{21}.
(ii) T_{21} and T_{41}.
(iii) T_{11} and T_{31}.

(3H)

(b) Find $\mathbb{E}(s^{T_{11}})$, and state whether or not T_{11} is defective. (5H)

9. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $\{1, 2, 3, 4\}$. The incomplete transition diagram is shown below. The diagram shows the states of the Markov chain, but arrows and probabilities are not marked.



Let $\boldsymbol{h} = \left(\frac{1}{2},\ 1,\ \frac{1}{4},\ 0\right)$ be the vector of hitting probabilities ***to*** one of the states on the diagram above, which we shall call state x. That is,

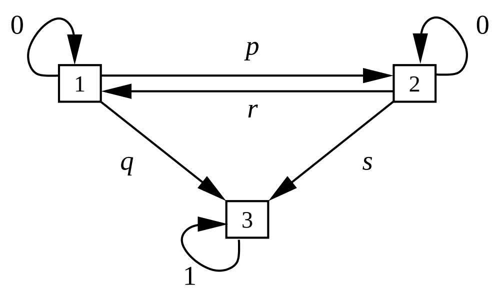
$$h_i = \mathbb{P}(\text{the chain hits state } x \text{ at any time } t \geq 0, \text{ starting from state } i), \quad \text{for } i = 1, 2, 3, 4.$$

(a) State x is either 1, 2, 3, or 4. Identify state x. (1M)

(b) Reproduce the diagram above and add arrows and probabilities to give a transition diagram that produces the vector $\boldsymbol{h}$ of hitting probabilities to state x. [Hint: there are many possible solutions; you are asked to produce just *one*. A simple diagram is sufficient.] (3H)

(c) What is the communicating class containing State 4 in your answer to part (b)? Explain why *every* possible solution to part (b) has the same communicating class for State 4. (2H)

(d) Now suppose $\{X_0, X_1, X_2, \ldots\}$ is a Markov chain on the state space $\{1, 2, 3\}$ with the transition diagram below.



Let $\boldsymbol{m} = (2,\ 3,\ 0)$ be the vector of mean hitting times ***to*** state 3. That is,

$$m_i = \mathbb{E}(\text{number of steps to hit state 3, starting from state } i), \quad \text{for } i = 1, 2, 3.$$

Find the missing probabilities p, q, r, and s, and draw the completed transition diagram. (4H)

10. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution. Define

$$P(s) = \mathbb{E}\left(s^Y\right), \qquad G_n(s) = \mathbb{E}\left(s^{Z_n}\right).$$

Note: we usually write $G(s)$ instead of $P(s)$, but we will need the notation $P(s)$ for this question.

(a) Using the fact that $Z_{n+1} = Y_1 + \ldots + Y_{Z_n}$ where each $Y_i \sim Y$ and all the Y_i's are independent, prove that

$$G_{n+1}(s) = G_n\Big(P(s)\Big).$$

(4M)

Now suppose that we have a branching process with additional immigration. For each generation n, the process continues as usual, with the addition of a random number of M_n immigrants that join the population from outside. Assume that M_n is independent of Z_{n-1} and of $Y_1, \ldots, Y_{Z_{n-1}}$ in any generation. Each of the M_n immigrants behaves exactly like any other individual: it reproduces in the following generation independently of all other individuals, with family size $\sim Y$. Under the immigration model, we have:

$$\begin{aligned} Z_1 &= Y_1 + M_1 \\ &\vdots \\ Z_n &= Y_1 + \ldots + Y_{Z_{n-1}} + M_n \\ Z_{n+1} &= Y_1 + \ldots + Y_{Z_n} + M_{n+1}, \end{aligned}$$

where $M_1, \ldots, M_{n+1}$ are the random number of immigrants that join the population at generations $1, 2, \ldots, n+1$.

Let $M_1, \ldots, M_{n+1}$ be independent and identically distributed with probability generating function $H(s) = \mathbb{E}(s^M)$. As before, let $P(s) = \mathbb{E}\left(s^Y\right)$ and $G_n(s) = \mathbb{E}\left(s^{Z_n}\right)$. Also, define $P_n(s)$ to be the n-fold iterate of P: that is,

$$P_n(s) = \underbrace{P\Bigg(P\Bigg(P\Bigg(\ldots P(s)}_{n \text{ times}} \ldots\Bigg)\Bigg)\Bigg).$$

We wish to prove by mathematical induction that

$$G_n(s) = H(s) \times H\Big(P_1(s)\Big) \times H\Big(P_2(s)\Big) \times \ldots \times H\Big(P_{n-1}(s)\Big) \times P_n(s). \qquad (\star)$$

(b) Prove that equation $(\star)$ holds when $n = 1$: that is, prove that

$$G_1(s) = \mathbb{E}\left(s^{Z_1}\right) = H(s)P_1(s).$$

(3H)

(c) Complete the proof of equation $(\star)$ by mathematical induction. (5H)

ATTACHMENT

1. **Discrete Probability Distributions**

Distribution	$\mathbb{P}(X=x)$	$\mathbb{E}(X)$	PGF, $\mathbb{E}(s^X)$
Geometric(p)	pq^x (where $q=1-p$), for $x=0,1,2,\ldots$	$\frac{q}{p}$	$\frac{p}{1-qs}$
	Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success})=p$.		
Binomial(n,p)	$\binom{n}{x}p^x q^{n-x}$ (where $q=1-p$), for $x=0,1,2,\ldots,n$.	np	$(ps+q)^n$
	Number of successes in n independent trials, each with $\mathbb{P}(\text{success})=p$.		
Poisson(λ)	$\frac{\lambda^x}{x!}e^{-\lambda}$ for $x=0,1,2,\ldots$	λ	$e^{\lambda(s-1)}$

2. **Uniform Distribution:** $X \sim \text{Uniform}(a,b)$.
Probability density function, $f_X(x)=\frac{1}{b-a}$ for $a<x<b$. Mean, $\mathbb{E}(X)=\frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

Definition:	$G_X(s)=\mathbb{E}(s^X)$	
Moments:	$\mathbb{E}(X)=G_X'(1)$	$\mathbb{E}\Big\{X(X-1)\ldots(X-k+1)\Big\}=G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X=n)=\frac{1}{n!}G_X^{(n)}(0)$	

4. **Geometric Series:** $1+r+r^2+r^3+\ldots \; = \; \sum_{x=0}^{\infty} r^x \; = \; \frac{1}{1-r}$ for $|r|<1$.

Finite sum: $\sum_{x=0}^{n} r^x = \frac{1-r^{n+1}}{1-r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p,q \in \mathbb{R}$, and integer $n>0$, $(p+q)^n \; = \; \sum_{x=0}^{n}\binom{n}{x}p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\sum_{x=0}^{\infty}\frac{\lambda^x}{x!}=e^{\lambda}$.