THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2006
Campus: City

STATISTICS

Stochastic Processes
Special Topic in Applied Probability

(Time allowed: THREE hours)

NOTE: Attempt **ALL** questions. Marks for each question are shown in brackets.
An **Attachment** containing useful information is found on page 7.
You may use information from the Attachment without derivation at any time.

1. Let A, B, and C be any events on a sample space Ω. For each of the following statements, say whether they are **true** or **false** in general. Show any working or explanations you use.

   (a) $\mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid B \cap C)\,\mathbb{P}(B \mid C)$. (2M)

   (b) $\mathbb{P}(A \mid B \cup C) = \mathbb{P}(A \mid B) + \mathbb{P}(A \mid C) - \mathbb{P}(A \mid B \cap C)$. (2M)

   (c) $\mathbb{P}(A \mid B) + \mathbb{P}(\overline{A} \mid B) = \mathbb{P}(A)$. (2M)

2. Let A and B be events with $\mathbb{P}(A) = \frac{7}{8}$ and $\mathbb{P}(B) = \frac{1}{4}$. Show that

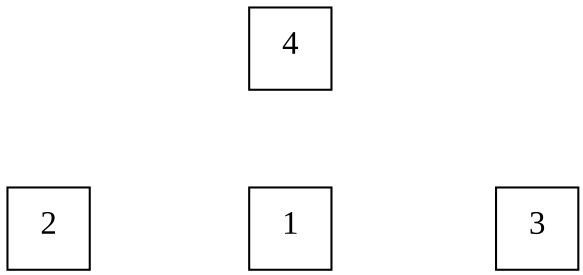
$$\frac{1}{8} \leq \mathbb{P}(A \cap B) \leq \frac{1}{4}.$$

   [Hint: consider $\mathbb{P}(A \cup B)$.] (4E)

3. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3, 4\}$, with transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Copy the diagram below, and add arrows and probabilities to complete the transition diagram. (2E)



(b) Identify all communicating classes. For each class, state whether or not it is closed. (2E)

(c) State whether the Markov chain is irreducible, and whether or not all states are aperiodic. (2E)

(d) By solving the equilibrium equations, find an equilibrium distribution $\boldsymbol{\pi}$ for this Markov chain. Does the Markov chain converge to this distribution as $t \to \infty$, regardless of its start state? (3M)

(e) Let $\boldsymbol{h_A} = (h_{1A}, h_{2A}, h_{3A}, h_{4A})$ be the vector of hitting probabilities to the set $A = \{2, 3\}$. Find the vector $\boldsymbol{h_A}$. (3E)

4. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2\}$, with transition matrix P. The general solution for P^t is as follows:

$$P^t = \frac{1}{4}\left\{ \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} + \left(\frac{1}{5}\right)^t \begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix} \right\} \qquad \text{for } t = 1, 2, 3, \ldots.$$

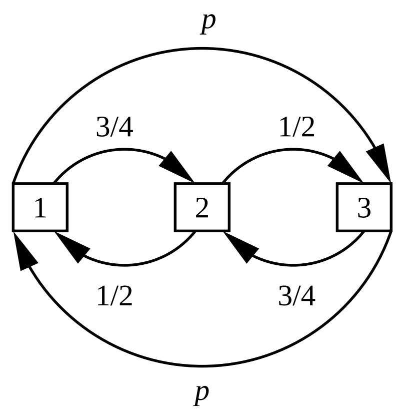
(a) Draw the transition diagram of the Markov chain. (2E)

(b) Suppose that $X_0 \sim \left(\frac{2}{3}, \frac{1}{3}\right)$. Find a vector describing the distribution of X_1. (2E)

(c) Again suppose that $X_0 \sim \left(\frac{2}{3}, \frac{1}{3}\right)$. Find $\mathbb{P}(X_0 = 2,\ X_2 = 2,\ X_5 = 1)$. (4M)

(d) Using *only* the formula given for P^t, state whether or not the Markov chain converges to an equilibrium distribution as $t \to \infty$, independent of the chain's start state. If the chain does converge to equilibrium, state what the equilibrium distribution is. If the chain does not converge to equilibrium, explain why. (3M)

5. Three friends stand in a row in the park and throw a ball to each other according to the following transition diagram.



Let T_{ij} be the number of times the ball is thrown until it first returns to person j, starting with person i. For example, if the ball is thrown person 1 $\to$ person 2 $\to$ person 1, then $T_{11} = 2$. If the ball is thrown person 2 $\to$ person 3, then $T_{23} = 1$.

(a) What is the missing probability p? (1E)

(b) For each of the following pairs of random variables, state whether or not the two random variables have the same distribution as each other.

(i) T_{11} and T_{33}.
(ii) T_{12} and T_{21}.
(iii) T_{12} and T_{32}. (3M)

(c) You may assume that $\mathbb{E}\left(s^{T_{11}}\right) = \dfrac{s^2(7+3s)}{2(8-3s^2)}$, and $\mathbb{E}\left(s^{T_{21}}\right) = \dfrac{s(4+s)}{8-3s^2}$.

Find $\mathbb{E}\left(s^{T_{31}}\right)$, simplifying so that your answer is expressed as a fraction with denominator $8-3s^2$. (3M)

(d) Define the random variable W to be the number of times the ball is thrown until the **second** time it reaches person 1, starting with person 2. For example, if the ball is thrown person 2 $\to$ person 1 $\to$ person 3 $\to$ person 1, then $W = 3$. Find $\mathbb{E}\left(s^W\right)$, fully justifying your working. (3H)

6. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and let γ be the probability of ultimate extinction.

Suppose that $Y \sim \text{Geometric}\,(p)$, so that

$$\mathbb{P}(Y = y) = pq^y \qquad \text{for } y = 0, 1, \ldots,$$

where $q = 1 - p$.

(a) Let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. Show that

$$G(s) = \frac{p}{1 - qs},$$

and state the radius of convergence. (3E)

(b) Use $G(s)$ to find $\mathbb{E}(Y)$ when $Y \sim \text{Geometric}(p)$. (3M)

(c) Suppose that $Y \sim \text{Geometric}\left(p = \frac{2}{5}\right)$. Show that

$$G(s) = \frac{2}{5 - 3s}.$$

(1E)

(d) Continue to suppose that $Y \sim \text{Geometric}\left(p = \frac{2}{5}\right)$. Let $G_2(s)$ be the probability generating function of Z_2. Write down an expression for $G_2(s)$, simplifying your answer as much as possible. (3E)

(e) Using your answers to the previous parts, find $\mathbb{P}(Z_1 = 0)$, $\mathbb{P}(Z_2 = 0)$, $\mathbb{P}(Z_3 = 0)$, and $\mathbb{P}(Z_4 = 0)$, when $Y \sim \text{Geometric}\left(\frac{2}{5}\right)$. (3M)

(f) Continue to suppose that $Y \sim \text{Geometric}\left(\frac{2}{5}\right)$. Find the probability of eventual extinction, γ. (3E)

(g) Now suppose that $Y \sim \text{Poisson}\left(\lambda = \frac{2}{5}\right)$. What is the probability of eventual extinction, γ? (2M)

7. Suppose that X and Y are discrete random variables satisfying:

$$\begin{aligned} X &\sim \text{Poisson}(\lambda) \\ [Y \mid X] &\sim \text{Binomial}(X, p). \end{aligned}$$

Find the probability generating function of Y, $G_Y(s) = \mathbb{E}(s^Y)$. Hence name the distribution of Y, fully specifying all parameters. (5H)

8. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, where

$$Y = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1 - p, \end{cases}$$

and where $0 < p < 1$.

Let $G(s) = \mathbb{E}\left(s^Y\right)$ be the probability generating function of Y, and let $G_n(s) = \mathbb{E}\left(s^{Z_n}\right)$ be the probability generating function of Z_n.

(a) Prove by mathematical induction that

$$G_n(s) = p^n s + 1 - p^n \quad \text{for } n = 1, 2, 3, \ldots$$

Marks will be given for setting out your answer clearly and logically. (6H)

(b) Show that the probability that the branching process becomes extinct *by* generation n is $1 - p^n$. Explain in words why the probability has this simple formula. (3M)

(c) Find the probability that the branching process becomes extinct *at* generation n. (2M)

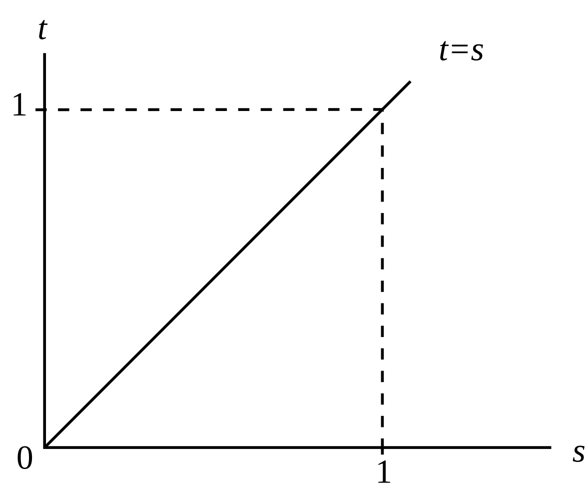
(d) Find the probability of ultimate extinction, γ. (2E)

(e) The diagram below shows a graph of $t = s$ for $0 \le s \le 1$. Copy the diagram and mark on it the following features specific to **this** branching process:

(i) The graph of $t = G(s)$;

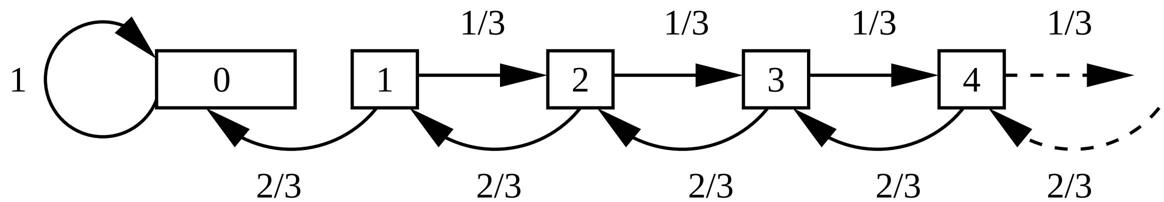
(ii) γ;

(iii) $\mathbb{P}(Y = 0)$. (3M)



9. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{0, 1, 2, 3, \ldots\}$, with the transition diagram below. You may assume that the pattern continues indefinitely, in other words that the transition probabilities are:

$$\begin{aligned} p_{i,i+1} &= \tfrac{1}{3} \text{ for all } i = 1, 2, 3, \ldots \\ p_{i,i-1} &= \tfrac{2}{3} \text{ for all } i = 1, 2, 3, \ldots \end{aligned}$$



Given a starting state k, define h_k to be the probability that the process eventually reaches state 0, starting from state k. We aim to prove by mathematical induction that

$$h_k = \left(2^k - 1\right) h_1 - \left(2^k - 2\right) \qquad \text{for } k = 0, 1, 2, \ldots, \qquad (\star)$$

where h_1 is the probability that the process eventually reaches state 0, starting from state 1.

(a) Show that formula $(\star)$ is true for $k = 0$ and $k = 1$. (2M)

(b) By considering the hitting probability equation for h_r, show that

$$h_{r+1} = 3h_r - 2h_{r-1} \qquad \text{for } r = 1, 2, 3, \ldots \qquad \text{(2M)}$$

(c) Suppose that formula $(\star)$ is true for all $k = 0, 1, \ldots, r$, for some integer $r \geq 1$. Using part (b), show that formula $(\star)$ is true for $k = r + 1$. Hence deduce that formula $(\star)$ is true for all $k = 0, 1, 2, \ldots$. (4H)

(d) It remains to find the value of h_1. Using the fact that $0 \leq h_k \leq 1$ for all k, find h_1. Hence find h_k for any $k = 0, 1, 2, \ldots$. (3H)

(e) Let T be a random variable giving the number of steps taken to reach state 0, starting from state 10. Is T a defective random variable? Explain your answer. (2H)

10. Let $T \sim \text{Exponential}(6)$ and let $W \sim \text{Exponential}(8)$, and let T and W be independent. Find $\mathbb{P}(W < T)$. (5H)

Hint: for this question, you may use the following information:

When $X \sim \text{Exponential}(\lambda)$, the probability density function of X is

$$f_X(x) = \lambda e^{-\lambda x} \text{ for } 0 < x < \infty,$$

and the cumulative distribution function of X is

$$F_X(x) = \mathbb{P}(X \leq x) = 1 - e^{-\lambda x} \text{ for } 0 < x < \infty.$$

ATTACHMENT

1. **Discrete Probability Distributions**

Distribution	$\mathbb{P}(X = x)$	$\mathbb{E}(X)$	PGF, $\mathbb{E}(s^X)$
Geometric(p)	pq^x (where $q = 1 - p$), for $x = 0, 1, 2, \ldots$	$\frac{q}{p}$	$\frac{p}{1 - qs}$
	Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Binomial(n, p)	$\binom{n}{x} p^x q^{n-x}$ (where $q = 1 - p$), for $x = 0, 1, 2, \ldots, n$.	np	$(ps + q)^n$
	Number of successes in n independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Poisson(λ)	$\frac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \ldots$	λ	$e^{\lambda(s-1)}$

2. **Uniform Distribution:** $X \sim \text{Uniform}(a, b)$.
Probability density function, $f_X(x) = \frac{1}{b-a}$ for $a < x < b$. Mean, $\mathbb{E}(X) = \frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

Definition:	$G_X(s) = \mathbb{E}(s^X)$	
Moments:	$\mathbb{E}(X) = G_X'(1)$	$\mathbb{E}\Big\{ X(X-1)\ldots(X-k+1) \Big\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \frac{1}{n!} G_X^{(n)}(0)$	

4. **Geometric Series:** $1 + r + r^2 + r^3 + \ldots = \sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$ for $|r| < 1$.

Finite sum: $\sum_{x=0}^{n} r^x = \frac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p, q \in \mathbb{R}$, and integer $n > 0$, $(p + q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$.