THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2008
Campus: City

STATISTICS

Stochastic Processes

(Time allowed: THREE hours)

NOTE: Attempt **ALL** questions. Marks for each question are shown in brackets.
An **Attachment** containing useful information is found on page 7.

CONTINUED

1. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3\}$, with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}.$$

   (a) Draw the transition diagram. (2E)

   (b) Identify all communicating classes. For each class, state whether or not it is closed. (2E)

   (c) Find an equilibrium distribution $\boldsymbol{\pi}$ for P. (4E)

   (d) Does X_t converge to the distribution in (c) as $t \to \infty$? Explain why or why not. (2M)

   (e) Let $\boldsymbol{m} = (m_1, m_2, m_3)$ be the vector of expected reaching times for reaching state 3. Find the vector $\boldsymbol{m}$. (3E)

2. Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed discrete random variables, with common probability generating function $G_X(s)$. Let N be a discrete random variable with probability generating function $G_N(s)$, where N is independent of $X_1, X_2, \ldots$.

   Let $T = X_1 + X_2 + \ldots + X_N$ be the randomly stopped sum of the X_i's, and let $G_T(s)$ be the probability generating function of T.

   (a) Show that $G_T(s) = G_N\Big(G_X(s)\Big)$. (4M)

   (b) For each i, let

$$X_i = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1 - p. \end{cases}$$

   Write down $G_X(s)$, the common probability generating function of $X_1, X_2, \ldots$. (2E)

   (c) Suppose that $N \sim \text{Poisson}(\lambda)$. Using the list of probability generating functions in the Attachment, find the probability generating function of $T = X_1 + \ldots + X_N$. Hence name the distribution of T. (3M)

3. Weetbix breakfast cereals are having a promotion. Every packet of Weetbix contains a discount card. The card is worth \$1, \$2, \$3, or \$4 with equal probability, independently of the cards in other packets. Cards are coloured red, yellow, green, or blue, according to their value. The \$1 cards are coloured red, the \$2 cards are coloured yellow, and so on.

   (a) Three different Markov chains $\{Y_0, Y_1, Y_2, Y_3, \ldots\}$ are defined below. In each case, we begin with 0 packets at time 0, and define $Y_0 = 0$. For each chain, write down the transition matrix. Label the state space in the margins of your matrix.

      (i) Y_t is the **maximum** card value obtained in the first t packets.

      (ii) Y_t is the **number of different colours** in the cards obtained from the first t packets.

      (iii) Y_t is the **number of blue cards** obtained in the first t packets. (9M)

   (b) Define T to be the number of packets required before all four colours have been collected. Find $\mathbb{E}(T)$. [Hint: one of your answers to part (a) will be helpful.] (5H)

4. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and suppose that $Y \sim \text{Binomial}(2, \frac{4}{5})$.

   (a) Let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. Working from the probability function of Y, show that

$$G(s) = \frac{1}{25}(4s + 1)^2 \qquad \text{for} \quad s \in \mathbb{R}.$$

   (3E)

   (b) Let $G_2(s)$ be the probability generating function of Z_2. Find $G_2(s)$. (Do **not** simplify the expression.) (2E)

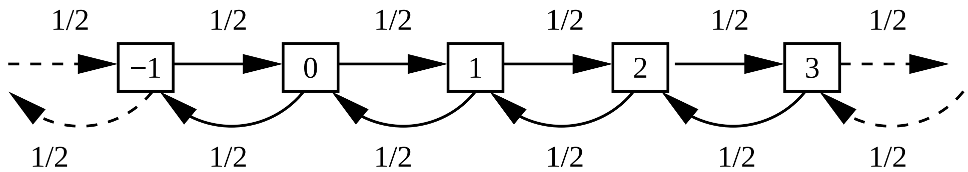
   (c) Show that $\mathbb{P}(Z_4 = 0) = 0.061$. (3M)

   (d) Find the probability of eventual extinction, γ. (3E)

   (e) Suppose that $Z_2 = 5$. Find the probability of eventual extinction. (2E)

   (f) Suppose again that $Z_2 = 5$. Using part (c), find the probability that all 5 of the individuals that were alive at generation 2 still have living descendents in generation 6. (3M)

5. Let $\{X_0, X_1, X_2, \ldots\}$ be a symmetric random walk on the integers, with transition diagram below.



Let T be the number of steps taken to reach state 1, starting at state 0. Let $H(s) = \mathbb{E}(s^T)$ be the probability generating function of T.

(a) Show that $H(s)$ must be either the (+) root or the (−) root of the following expression:

$$H(s) = \frac{1 \pm \sqrt{1 - s^2}}{s}\,. \qquad \text{(4M)}$$

(b) By considering $H(0)$, prove that $H(s)$ can **not** be the (+) root in the expression above. (2M)

(c) Using the expression $H(s) = \dfrac{1 - \sqrt{1 - s^2}}{s}$, find whether T is a defective random variable. (2M)

(d) Using $H(s)$, find $\mathbb{E}(T)$. (3M)

(e) What is the probability that we *never* reach state 1, starting from state 0? (1M)

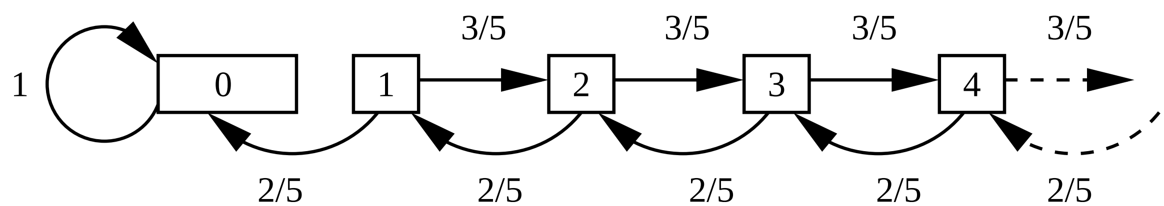
(f) Let $p_t = \mathbb{P}(T = t)$ for $t = 0, 1, 2, 3, \ldots$. State the values of p_0, p_1, and p_2, and show that

$$p_t = \frac{1}{2} \sum_{x=0}^{t-1} p_x \, p_{t-1-x}$$

for all $t = 3, 4, 5, \ldots$. (6H)

6. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{0, 1, 2, 3, \ldots\}$, with the transition diagram below. You may assume that the pattern continues indefinitely, in other words that the transition probabilities are:

$$\begin{aligned} p_{i,i+1} &= \tfrac{3}{5} \text{ for all } i = 1, 2, 3, \ldots \\ p_{i,i-1} &= \tfrac{2}{5} \text{ for all } i = 1, 2, 3, \ldots \end{aligned}$$



For any state x, define h_x to be the probability that the process eventually reaches state 0, starting from state x.

(a) Using first-step analysis, show that

$$3h_{x+1} - 5h_x + 2h_{x-1} = 0 \qquad \text{for } x = 1, 2, 3, \ldots, \qquad (\star)$$

and state the boundary condition for h_0. (3E)

(b) The general solution to the difference equation $(\star)$ is

$$h_x = At_1^x + Bt_2^x \quad \text{for } x = 0, 1, 2, 3, \ldots$$

where A and B are constants to be found, and t_1 and t_2 are the two roots of the following quadratic equation:

$$3t^2 - 5t + 2 = 0.$$

Find the roots t_1 and t_2 of the quadratic equation, and thus state the general solution of $(\star)$ in terms of the unknown constants A and B. Hence use the boundary condition for h_0 to show that

$$h_x = 1 - A\left\{1 - \left(\frac{2}{3}\right)^x\right\}, \quad \text{for } x = 0, 1, 2, 3, \ldots$$

where A is an unknown constant. (4M)

(c) Using the theorem that the hitting probabilities $(h_0, h_1, h_2, \ldots)$ are the minimal non-negative solution to the hitting probability equations, find A. Hence give a formula for h_x for all $x = 0, 1, 2, \ldots$. (4H)

(d) Would it be possible to prove your formula in (c) by mathematical induction? Explain why or why not. (2H)

(e) Let T be a random variable giving the number of steps taken to reach state 0, starting from state 1. Is T a defective random variable? Explain your answer. (2H)

7. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, where $Y \sim \text{Geometric}\,(p)$ and $\mathbb{E}(Y) = \mu = (1-p)/p$.

   Let $G_n(s) = \mathbb{E}(s^{Z_n})$ be the probability generating function of Z_n, and let $G(s) = G_1(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. You may assume that

$$G(s) = \frac{1}{\mu + 1 - \mu s}\,.$$

   Define

$$\gamma_n = \mathbb{P}(\text{extinction by generation } n) = \mathbb{P}(Z_n = 0),$$

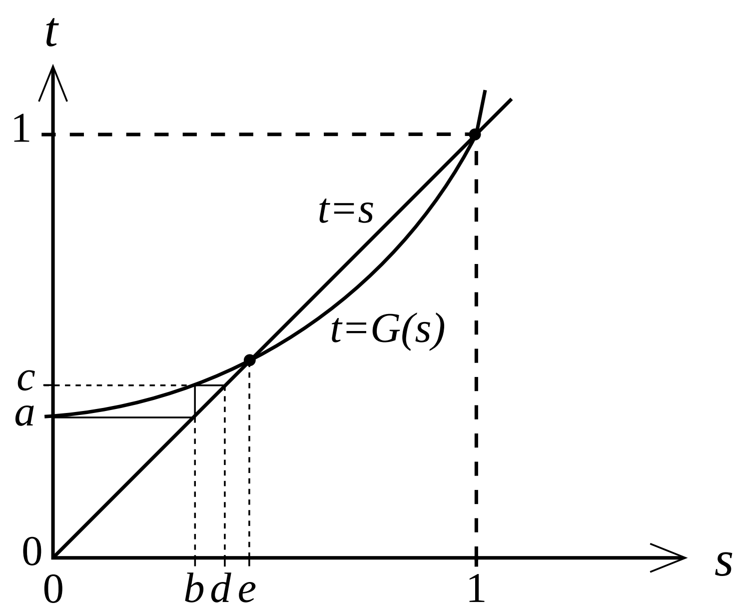
   and let γ be the probability of ultimate extinction. You may assume that

$$\gamma_n = G_n(0).$$

   (a) The diagram below shows a graph of $G(s)$, with missing values a, b, c, d, and e. Each of the missing values is included on the following list:

$$0 \qquad 1 \qquad \mu \qquad \sigma^2 \qquad \gamma \qquad \gamma_0 \qquad \gamma_1 \qquad \gamma_2 \qquad \gamma_3$$

   By reference to the list above, state the values of a, b, c, d, and e. (5M)



   (b) State whether $\mu < 1$, $\mu = 1$, or $\mu > 1$ in the diagram above, and explain why. (2M)

   (c) We wish to prove by mathematical induction that

$$\gamma_n = \frac{\mu^n - 1}{\mu^{n+1} - 1} \qquad \text{for } n = 1, 2, 3, \ldots, \qquad (\star)$$

   for any μ with $\mu \neq 1$. Prove $(\star)$ by mathematical induction. You may assume without proof the branching process recursion formula. Marks will be given for good layout of your answer. (8H)

ATTACHMENT FOLLOWS

ATTACHMENT

1. **Discrete Probability Distributions**

Distribution	$\mathbb{P}(X = x)$	$\mathbb{E}(X)$	PGF, $\mathbb{E}(s^X)$
Geometric(p)	pq^x (where $q = 1 - p$), for $x = 0, 1, 2, \ldots$	$\dfrac{q}{p}$	$\dfrac{p}{1 - qs}$
	Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Binomial(n, p)	$\binom{n}{x} p^x q^{n-x}$ (where $q = 1 - p$), for $x = 0, 1, 2, \ldots, n$.	np	$(ps + q)^n$
	Number of successes in n independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Poisson(λ)	$\dfrac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \ldots$	λ	$e^{\lambda(s-1)}$

2. **Uniform Distribution:** $X \sim \text{Uniform}(a, b)$.
Probability density function, $f_X(x) = \frac{1}{b-a}$ for $a < x < b$. Mean, $\mathbb{E}(X) = \frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

Definition:	$G_X(s) = \mathbb{E}(s^X)$	
Moments:	$\mathbb{E}(X) = G'_X(1)$	$\mathbb{E}\left\{X(X-1)\ldots(X-k+1)\right\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \dfrac{1}{n!} G_X^{(n)}(0)$	

4. **Geometric Series:** $1 + r + r^2 + r^3 + \ldots \;=\; \sum_{x=0}^{\infty} r^x \;=\; \dfrac{1}{1-r}$ for $|r| < 1$.

Finite sum: $\displaystyle\sum_{x=0}^{n} r^x = \frac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p, q \in \mathbb{R}$, and integer $n > 0$, $(p+q)^n \;=\; \displaystyle\sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\displaystyle\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$.