THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2009
Campus: City

STATISTICS

Stochastic Processes

(Time allowed: THREE hours)

NOTE: Attempt **ALL** questions. Marks for each question are shown in brackets.
An **Attachment** containing useful information is found on page 9.

1. Let $\{X_1, X_2, X_3, \ldots\}$ be a Markov chain on the state space $S = \{1, 2\}$, with transition matrix

$$P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}.$$

Assume that $0 < p < 1$ and $0 < q < 1$.

(a) Draw the transition diagram. (2E)

(b) Identify all communicating classes. For each class, state whether or not it is closed. (1E)

(c) Show that P has the equilibrium distribution

$$\boldsymbol{\pi} = \left(\frac{1-q}{2-p-q}, \quad \frac{1-p}{2-p-q} \right).$$

(4M)

(d) Does X_t converge to the distribution in (c) as $t \to \infty$? Explain why or why not. (2M)

Ron is playing a game against a computer. On each turn, the computer will select either a *red* or a *blue* card with equal probability 0.5. Ron will also select a red card or a blue card: however, he selects red with probability r, and blue with probability $1-r$. All selections are independent on every turn. The rules of the game are as follows:

- If Ron's card and the computer's card are of different colours, the winner of the turn is the holder of the *red* card.
- If Ron's card and the computer's card are both of the same colour, the winner of the turn is the *loser* of the *previous* turn.

Let $\{X_1, X_2, X_3, \ldots\}$ be a Markov chain denoting the state of the game at turn t, where the state of the game is either *'Ron Loses'* or *'Ron Wins'* at turn t. The incomplete transition diagram is below.

Ron Loses	**Ron Wins**

(e) Copy down the diagram, and add all arrows and probabilities. Be careful to ensure that arrows leading out of every box sum to one. (5M)

(f) Using the expression in part (c), find the equilibrium distribution $\boldsymbol{\pi}$ for Ron's Markov chain, in terms of Ron's probability r. (3E)

(g) Explain why the long-term proportion of time the Markov chain spends in state *'Ron Wins'* is

$$\frac{1+r}{3}.$$

(2M)

(h) Using the expression in (g), what value of r should Ron choose if he wishes to spend the greatest possible proportion of time in the state *'Ron Wins'*? (1E)

[20 marks]

2. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and suppose that $Y \sim \text{Binomial}(3, \frac{1}{2})$.

   (a) Let $G(s) = \mathbb{E}(s^Y)$ be the probability generating function of Y. Working from the probability function of Y, show that

   $$G(s) = \tfrac{1}{8}\left(s^3 + 3s^2 + 3s + 1\right) \qquad \text{for } s \in \mathbb{R}.$$

   (3E)

   (b) Show that $\mathbb{P}(Z_2 = 0) = 0.178$. (2E)

   (c) What is the probability that the process is *not* extinct by generation 2? (1E)

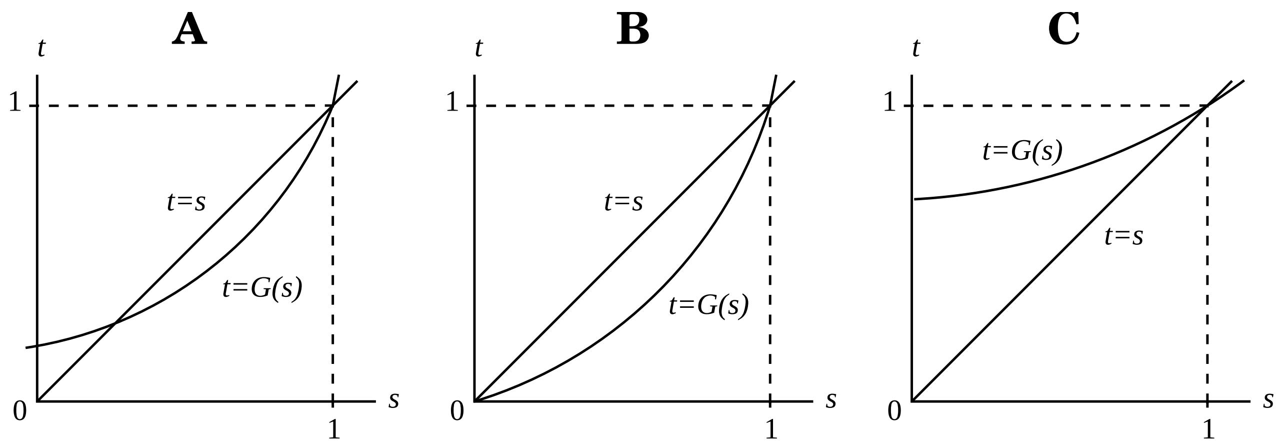
   (d) Find the probability of eventual extinction, γ. (4M)

   (e) Suppose that $Z_2 = 3$. Find the probability of eventual extinction. (2E)

[12 marks]

3. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and let $G(s) = \mathbb{E}\left(s^Y\right)$ be the probability generating function of Y. Let $\mu = \mathbb{E}(Y)$ be the mean family size.

   The diagram below shows graphs of $t = s$ and $t = G(s)$ for $0 \le s \le 1$, for three branching processes $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, with different characteristics.



   (a) For which process or processes out of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ is $\mu < 1$? Briefly explain your answer. (2E)

   (b) For which process or processes is $\mu > 1$? Briefly explain your answer. (2E)

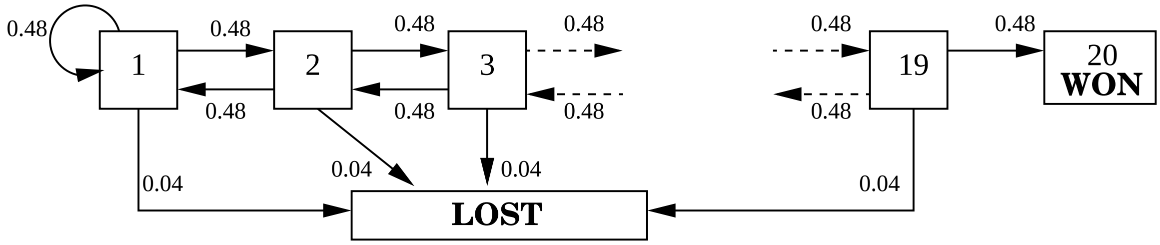
   (c) Let γ be the probability of ultimate extinction. For which process or processes is $\gamma = 1$? Briefly explain your answer. (2E)

   (d) For which process or processes is $\mathbb{P}(Y > 0) = 1$? Briefly explain your answer. (2M)

   (e) For which process or processes is $G(1) = 1$? Briefly explain your answer. (2M)

[10 marks]

4. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with the transition diagram below.



For any state x, let h_x be the probability that the process eventually reaches state 20 (**Won**), starting from state x.

(a) Using first-step analysis, show that

$$0.48h_{x+1} - h_x + 0.48h_{x-1} = 0 \qquad \text{for } x = 2, 3, 4, \ldots, 19 \qquad (\star)$$

(2E)

(b) Give boundary conditions for h_{20} and (as far as possible) for h_1. (3M)

(c) The general solution to the difference equation $(\star)$ is

$$h_x = At_1^x + Bt_2^x \quad \text{for } x = 1, 2, 3, \ldots, 20,$$

where A and B are constants to be found, and t_1 and t_2 are the two roots of the following quadratic equation:

$$0.48t^2 - t + 0.48 = 0.$$

Find the roots t_1 and t_2 of the quadratic equation. Assign t_1 to be the ***smaller*** of the roots, and t_2 to be the ***larger*** of the roots. Thus state the general solution of $(\star)$ in terms of the unknown constants A and B. (3E)

(d) Using one of the boundary conditions you stated in part (b), show that $B = \frac{3}{4}A$. (**Note:** if you have difficulty, make sure you have selected t_1 to be the ***smaller*** root in part (c).) (3M)

(e) Using part (d) and the other boundary condition you stated in part (b), show that the solution to $(\star)$ is

$$h_x = \frac{1}{(3/4)^{20} + (4/3)^{19}} \left\{ \left(\frac{3}{4}\right)^x + \left(\frac{4}{3}\right)^{x-1} \right\} \qquad \text{for } x = 1, \ldots, 20.$$

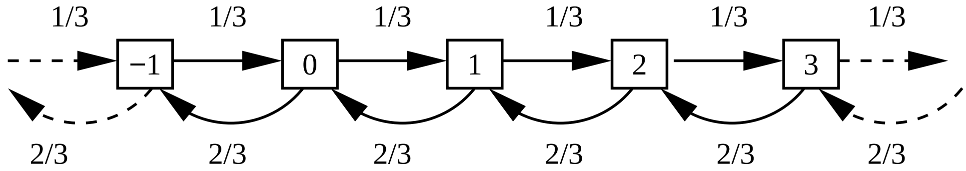
(3M)

(f) What is the probability that the process finishes in state **Lost**, starting from state 1? (2M)

(g) Let T be a random variable giving the number of steps taken to reach state 20, starting from state 1. Is T a defective random variable? Explain your answer. (2H)

[18 marks]

5. Let $\{X_0, X_1, X_2, \ldots\}$ be a random walk on the integers, with transition diagram below.



Let T be the number of steps taken to reach state 1, starting at state 0. Let $H(s) = \mathbb{E}(s^T)$ be the probability generating function of T.

(a) Show that $H(s)$ must be either the (+) root or the (−) root of the following expression:

$$H(s) = \frac{3 \pm \sqrt{9 - 8s^2}}{4s}\,.$$ (4M)

(b) By considering $\lim_{s\to 0} H(s)$, prove that $H(s)$ can **not** be the (+) root in the expression above. (2M)

(c) Using the expression $H(s) = \dfrac{3 - \sqrt{9 - 8s^2}}{4s}$, find whether T is a defective random variable. (2M)

(d) Let p_n be the probability that we *never* reach state n, starting from state 0, for $n = 1, 2, 3, \ldots$. Show that $p_1 = 1/2$. (1M)

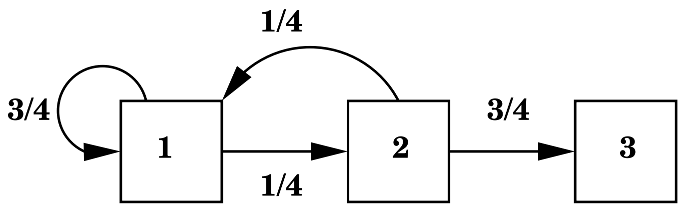
(e) We wish to prove that the probability we *never* reach state n, starting from state 0, is

$$p_n = 1 - \left(\frac{1}{2}\right)^n \qquad \text{for } n = 1, 2, 3, \ldots \qquad (\star)$$

Prove $(\star)$, either by mathematical induction or otherwise. You may use information from the Attachment in your proof if you wish. Marks will be given for clear explanation and layout of your answer. (5H)

[14 marks]

6. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $\{1, 2, 3\}$ with transition diagram shown below.



(a) Let $\boldsymbol{m} = (m_1, m_2, m_3)$ be the vector of expected reaching times for reaching state 3.
Show that $\boldsymbol{m} = \left(\frac{20}{3}, \frac{8}{3}, 0\right)$. (4E)

(b) Define random variables T and U as follows:

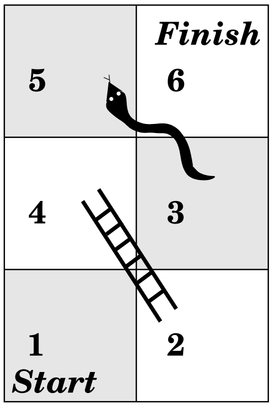
$$\begin{aligned} T &= \text{number of steps taken to hit state 3, starting from state 1.} \\ U &= \text{number of steps taken to hit state 3, starting from state 2.} \end{aligned}$$

(Note that T and U each count the number of *arrows* traversed before reaching state 3, not the number of *boxes* entered.)

Find $\mathbb{E}(T^2)$ and $\text{Var}(T)$. You may use any method you like, but try to avoid a method that requires differentiation if you can see how. (8H)

[12 marks]

7. A game of Snakes and Ladders is played on the following board of six squares:



The rules of the game are as follows.

- Squares are numbered 1, 2, . . . , 6.
- Start on Square 1 before the game begins.
- For each turn, **toss a fair coin.**
- If the toss is a ***head***, move forward ***one*** square. (For example, move to Square 2 if this is the first toss.)
- If the toss is a ***tail***, move forward ***two*** squares. (For example, move to Square 3 if this is the first toss.)
- If you land at the bottom of the ladder (Square 2), climb the ladder immediately. The turn finishes on the square at the top of the ladder (Square 4).
- If you land at the head of the snake (Square 5), slide down the snake immediately. The turn finishes on the square at the end of the snake's tail (Square 3).
- The game finishes when you first reach Square 6.

(a) Let T be the number of turns (tosses) taken for a player to finish the game, starting from Square 1. Find $\mathbb{E}(T)$. Define in words any notation you use. (6M)

(b) Let X be a random variable taking values $1, 2, 3, \ldots$, and suppose $\mathbb{E}(X) < \infty$. Show that

$$\mathbb{E}(X) = \sum_{x=1}^{\infty} \mathbb{P}(X \geq x).$$

[Hint: your answer should provide a clear but brief argument. Extensive calculations or proofs are not required.] (4H)

(c) Now suppose there are *two* players in the Snakes and Ladders game above. On each turn, both players toss a coin independently of each other, and make their respective moves. The game now finishes as soon as one of the players lands on Square 6. Let X be the number of turns taken for the game to finish. Show that

$$\mathbb{E}(X) = \sum_{t=1}^{\infty} \{\mathbb{P}(T \geq t)\}^2 ,$$

where T is the random variable defined in part (a). (4H)

[14 marks]

ATTACHMENT

1. **Discrete Probability Distributions**

Distribution	$\mathbb{P}(X = x)$	$\mathbb{E}(X)$	PGF, $\mathbb{E}(s^X)$
Geometric(p)	pq^x (where $q = 1 - p$), for $x = 0, 1, 2, \ldots$	$\frac{q}{p}$	$\frac{p}{1 - qs}$
	Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Binomial(n, p)	$\binom{n}{x} p^x q^{n-x}$ (where $q = 1 - p$), for $x = 0, 1, 2, \ldots, n$.	np	$(ps + q)^n$
	Number of successes in n independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Poisson(λ)	$\frac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \ldots$	λ	$e^{\lambda(s-1)}$

2. **Uniform Distribution:** $X \sim \text{Uniform}(a, b)$.
Probability density function, $f_X(x) = \frac{1}{b-a}$ for $a < x < b$. Mean, $\mathbb{E}(X) = \frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

Definition:	$G_X(s) = \mathbb{E}(s^X)$	
Moments:	$\mathbb{E}(X) = G'_X(1)$	$\mathbb{E}\Big\{X(X-1)\ldots(X-k+1)\Big\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \frac{1}{n!} G_X^{(n)}(0)$	

4. **Geometric Series:** $1 + r + r^2 + r^3 + \ldots \;=\; \sum_{x=0}^{\infty} r^x \;=\; \frac{1}{1-r}$ for $|r| < 1$.

Finite sum: $\sum_{x=0}^{n} r^x = \frac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p, q \in \mathbb{R}$, and integer $n > 0$, $(p + q)^n \;=\; \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$.