THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2010
Campus: City

STATISTICS

Stochastic Processes

(Time allowed: THREE hours)

NOTE: Attempt **ALL** questions. Marks for each question are shown in brackets.
There are 100 marks in total.
An **Attachment** containing useful information is found on page 8.

1. Let $\{X_1, X_2, X_3, \ldots\}$ be a Markov chain on the state space $S = \{1, 2, 3\}$, with transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1-\alpha & \alpha \\ 1-\alpha & \alpha & 0 \end{pmatrix}.$$

Assume that $0 < \alpha < 1$.

(a) Draw the transition diagram. (2E)

(b) Suppose the chain is equally likely to be in each of the three states at time 1.
Find $\mathbb{P}(X_1 = 2, X_2 = 3, X_3 = 1)$. (2E)

(c) By direct substitution, verify that P has equilibrium distribution

$$\boldsymbol{\pi}^T = \left(\frac{1-\alpha}{3-\alpha}, \quad \frac{1}{3-\alpha}, \quad \frac{1}{3-\alpha} \right).$$

(You do **not** need to find $\boldsymbol{\pi}$ by setting up and solving the equations.) (3E)

(d) Does X_t converge to an equilibrium distribution as $t \to \infty$? Explain why or why not. (2M)

[9 marks]

2. A doctor is trialling two different treatments for a disease: treatment A and treatment B. He uses a 'two-armed bandit' strategy:

- Patients are treated in succession, and are given labels 1, 2, 3, ...
- Patient 1 gets treatment A.
- If the treatment of Patient n is *successful*, then Patient $n+1$ gets the *same* treatment as Patient n (whichever treatment this is).
- If the treatment of Patient n is *unsuccessful*, then Patient $n+1$ gets the *other* treatment.

For any patient, the probability of *success* is α for treatment A, and β for treatment B. All patients are independent.

Define a Markov chain with state space $\{(A, S), (A, F), (B, S), (B, F)\}$, where state (A, S) means that the current patient is given treatment A and it is successful; state (A, F) means that the current patient is given treatment A and it fails; and similarly for (B, S) and (B, F).

(a) Write down the transition matrix for this Markov chain, and draw the transition diagram.
For the matrix, keep the states in the order $\{(A, S), (A, F), (B, S), (B, F)\}$. (6M)

(b) Find the equilibrium distribution, $\boldsymbol{\pi}$. (5M)

(c) Show that the long-run probability of *success* for each patient, using this strategy, is

$$\frac{\alpha + \beta - 2\alpha\beta}{2 - \alpha - \beta}.$$

(2M)

[13 marks]

3. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution. Suppose that the probability generating function of Y is

$$G(s) = \mathbb{E}(s^Y) = \frac{1}{10}\left(2 + 5s + 3s^2\right).$$

(a) Find $G'(s)$ and $G''(s)$. Hence, or otherwise, show that the probability function of Y is: (4E)

y	0	1	2
$\mathbb{P}(Y = y)$	0.2	0.5	0.3

(b) Using $G(s)$, show that $\mathbb{P}(Z_2 = 0) = 0.312$. (2E)

(c) Find $\mathbb{P}(Z_2 = 0)$ by an alternative method, using the Partition Theorem and partitioning over the possible values of Z_1. Show that you get the same answer as given in part (b). (3E)

(d) Find the probability of eventual extinction, γ. (3E)

(e) Suppose that $Z_2 = 3$. Find the probability of eventual extinction. (2E)

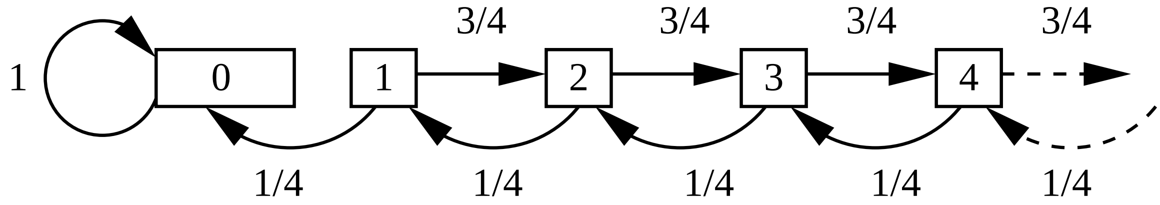
(f) Let T be the generation at which extinction occurs. Say whether T is a defective random variable, and give the value of $\mathbb{P}(T = \infty)$. (2M)

(g) What is $\mathbb{P}(T > 2)$? (2M)

(h) Now define a new branching process with family size distribution $X \sim Y + 3$. What is the probability of eventual extinction in this branching process? (2M)

[20 marks]

4. Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on the state space $S = \{0, 1, 2, 3, \ldots\}$, with the transition diagram below.



For any state x, define h_x to be the probability that the process eventually reaches state 0, starting from state x.

(a) What is $\mathbb{P}(X_3 = 0 \,|\, X_0 = 3)$? (1E)

(b) Show that

$$3h_{x+1} - 4h_x + h_{x-1} = 0 \qquad \text{for } x = 1, 2, 3, \ldots, \qquad (\star)$$

and state the boundary condition for h_0. (3E)

(c) The general solution to the difference equation $(\star)$ is

$$h_x = At_1^x + Bt_2^x \quad \text{for } x = 0, 1, 2, 3, \ldots$$

where A and B are constants to be found, and t_1 and t_2 are the two roots of the following quadratic equation:

$$3t^2 - 4t + 1 = 0.$$

Find the roots t_1 and t_2 of the quadratic equation. Assign t_1 to be the ***smaller*** of the roots, and t_2 to be the ***larger*** of the roots. State the general solution of $(\star)$ in terms of the unknown constants A and B, and hence use the boundary condition for h_0 to show that

$$h_x = 1 - A\left\{1 - \left(\frac{1}{3}\right)^x\right\}, \quad \text{for } x = 0, 1, 2, 3, \ldots.$$

(5M)

(d) Using the theorem that the hitting probabilities $(h_0, h_1, h_2, \ldots)$ are the minimal non-negative solution to the hitting probability equations, find A. Hence give a formula for h_x for all $x = 0, 1, 2, \ldots$. (4M)

[13 marks]

5. Let $X \sim \text{Geometric}(\alpha)$, where $0 < \alpha < 1$. The probability function of X is

$$\mathbb{P}(X = x) = \alpha(1 - \alpha)^x, \text{ for } x = 0, 1, 2, \ldots.$$

(a) Working directly from the probability function of X, show that the probability generating function (PGF) of X is

$$G_X(t) = \mathbb{E}(t^X) = \frac{\alpha}{1 - (1 - \alpha)t}.$$

(3E)

(b) Let $Y \sim \text{Binomial}(n, p)$, where $0 < p < 1$. Working directly from the probability function of Y, show that the PGF of Y is

$$G_Y(s) = \mathbb{E}(s^Y) = (ps + 1 - p)^n .$$

(3E)

(c) Now suppose that X is Geometric as above, and that the **conditional** distribution of Y, given X, is Binomial:

$$X \sim \text{Geometric}(\alpha), \qquad [\,Y \mid X\,] \sim \text{Binomial}(X, p).$$

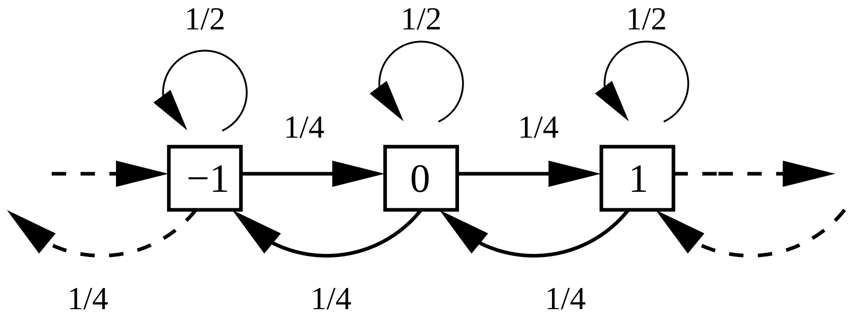
Show that

$$\mathbb{E}(s^Y) = \frac{\alpha}{\alpha + p(1 - \alpha) - p(1 - \alpha)s}.$$

Hence name the distribution of Y, and specify its parameters. (5H)

[11 marks]

6. Let $\{X_0, X_1, X_2, \ldots\}$ be a random walk on the integers, with transition diagram below.



Let U be the number of steps taken to reach state 1, starting at state 0. Let $H_U(s) = \mathbb{E}(s^U)$ be the probability generating function of U.

(a) Show that $H_U(s)$ must be either the (+) root or the (−) root of the following expression:

$$H_U(s) = \frac{2 - s \pm 2\sqrt{1-s}}{s}.$$ (4M)

(b) By considering $\lim_{s\to 0} H_U(s)$, prove that $H_U(s)$ can **not** be the (+) root in the expression above. (2M)

(c) Using the expression $H_U(s) = \dfrac{2 - s - 2\sqrt{1-s}}{s}$, find whether U is a defective random variable. (2M)

(d) Let V be the number of steps taken to reach state -1, starting at state 0. Let $H_V(s) = \mathbb{E}(s^V)$ be the probability generating function of V. Explain why $H_V(\cdot) = H_U(\cdot)$. (1M)

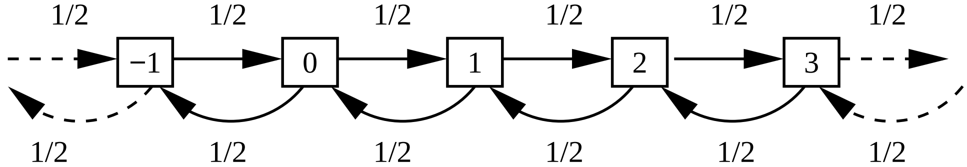
(e) Let T be the number of steps taken to **first return to** state 0, starting at state 0. For example, if $X_0 = 0$, $X_1 = 1$, and $X_2 = 0$, then $T = 2$ steps are taken to return from 0 to 0 again. Let $G(s) = \mathbb{E}(s^T)$ be the probability generating function of T. Show that

$$G(s) = 1 - \sqrt{1-s}.$$

(4M)

(f) Find the expected return time, $\mathbb{E}(T)$. (3M)

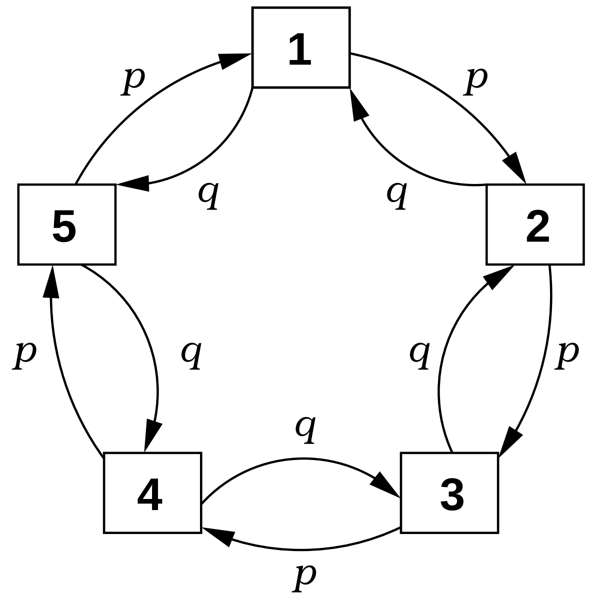
(g) Now suppose that $\{Y_0, Y_1, Y_2, \ldots\}$ and $\{Z_0, Z_1, Z_2, \ldots\}$ are **independent** random walks on the integers, both with the same transition diagram:



Suppose that the two random walks both start in state 0, that is, $Y_0 = Z_0 = 0$. We say that the two random walks *meet* if they are in the same state at some time. For example, the two walks meet at time 0 by definition, and they meet at time t if $Y_t = Z_t$. Let W be the number of steps taken before the two random walks first meet again after time 0. Find the probability generating function of W, $\mathbb{E}(s^W)$, and state whether or not the two random walks will certainly meet again. (6H)

[22 marks]

7. Consider the stochastic process with transition diagram shown below, where $q = 1 - p$. Define an **excursion from state 5 to state 2** as a path through the system, starting at state 5, and finishing when it reaches state 2 for the first time. The excursion **visits** a state x if the process is in state x at any time during the excursion. Every excursion from state 5 to state 2 visits both of the states 5 and 2.



Note: marks are awarded for formulating clear notation as well as for finding correct solutions.

(a) Find the probability that an excursion from state 5 to state 2:

(i) visits state 1;

(ii) visits state 3.

Evaluate both of these probabilities when $p = q = 0.5$. (8H)

(b) Find the probability that an excursion from state 5 to state 2 visits **all states**, in terms of p and q. You do **not** need to simplify your answer. Again, evaluate the probability when $p = q = 0.5$.
[Hint: this question requires thought but very little calculation.] (4H)

[12 marks]

ATTACHMENT

1. **Discrete Probability Distributions**

Distribution	$\mathbb{P}(X = x)$	$\mathbb{E}(X)$	PGF, $\mathbb{E}(s^X)$
Geometric(p)	pq^x (where $q = 1 - p$), for $x = 0, 1, 2, \dots$	$\dfrac{q}{p}$	$\dfrac{p}{1 - qs}$
	Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Binomial(n, p)	$\binom{n}{x} p^x q^{n-x}$ (where $q = 1 - p$), for $x = 0, 1, 2, \dots, n$.	np	$(ps + q)^n$
	Number of successes in n independent trials, each with $\mathbb{P}(\text{success}) = p$.		
Poisson(λ)	$\dfrac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \dots$	λ	$e^{\lambda(s-1)}$

2. **Uniform Distribution:** $X \sim \text{Uniform}(a, b)$.
Probability density function, $f_X(x) = \frac{1}{b-a}$ for $a < x < b$. Mean, $\mathbb{E}(X) = \frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

Definition:	$G_X(s) = \mathbb{E}(s^X)$	
Moments:	$\mathbb{E}(X) = G'_X(1)$	$\mathbb{E}\Big\{X(X-1)\dots(X-k+1)\Big\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \frac{1}{n!} G_X^{(n)}(0)$	

4. **Geometric Series:** $1 + r + r^2 + r^3 + \dots \;=\; \sum_{x=0}^{\infty} r^x \;=\; \dfrac{1}{1-r}$ for $|r| < 1$.

Finite sum: $\sum_{x=0}^{n} r^x = \dfrac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p, q \in \mathbb{R}$, and integer $n > 0$, $(p+q)^n \;=\; \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\sum_{x=0}^{\infty} \dfrac{\lambda^x}{x!} = e^{\lambda}$.