THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2014
Campus: City

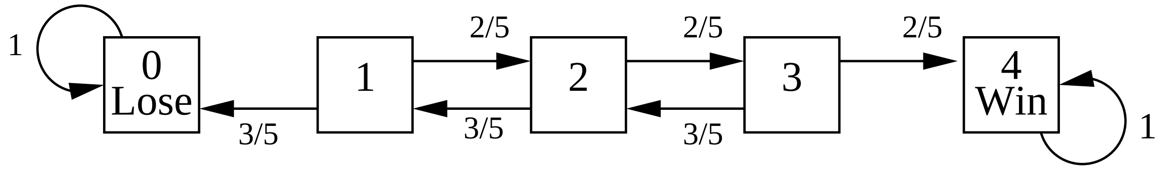
STATISTICS

Stochastic Processes

(Time allowed: THREE hours)

NOTE: Attempt **ALL** questions. Marks for each question are shown in brackets.
There are 100 marks in total.
An **Attachment** containing useful information is found on page 8.

1. A certain game corresponds to a Markov chain $\{X_0, X_1, X_2, \ldots\}$, following the transition diagram below.



Define the following events:

$$W = \{\text{process ever reaches state } 4\}\,;$$
$$D = \{\text{process ever reaches state } 1\}\,;$$
$$S_x = \{\text{process starts in state } x\} \quad \text{for } x = 0, 1, 2, 3, 4.$$

Define the following sets of hitting probabilities:

$$h_x = \mathbb{P}(W \mid S_x) \quad \text{for } x = 0, 1, 2, 3, 4\,;$$
$$d_x = \mathbb{P}(D \mid S_x) \quad \text{for } x = 2, 3, 4\,.$$

(a) Using first-step analysis, show that $d_2 = \frac{15}{19}$. You should show all working and equations. (3E)

(b) Write down the first-step analysis equations needed to find the vector $\boldsymbol{h} = (h_0, h_1, h_2, h_3, h_4)$. You do *not* need to solve the equations. (4E)

(c) The solution to the equations in part (b) is:

$$\boldsymbol{h} = \left(0\,,\ \tfrac{8}{65}\,,\ \tfrac{20}{65}\,,\ \tfrac{38}{65}\,,\ 1\right)\,.$$

Using this, and any other results you need from parts (a) and (b), find $\mathbb{P}(D \mid S_2 \cap W)$. Show all your working. Say which probability is larger, out of $\mathbb{P}(D \mid S_2 \cap W)$ and $\mathbb{P}(D \mid S_2)$, and briefly comment on why you would expect this to be the case. (5M)

(d) Write down all communicating classes of the Markov chain $\{X_0, X_1, X_2, \ldots\}$ denoted by the diagram above. For each class, say whether or not it is closed. (2E)

(e) Does the Markov chain $\{X_t\}$ converge to an equilibrium distribution that does not depend upon its start state as $t \to \infty$? Explain why or why not. (2E)

[16 marks]

2. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let Y be the family size distribution, and suppose that the probability generating function (PGF) of Y is:

$$G(s) = \mathbb{E}(s^Y) = \frac{1}{1 + \mu - \mu s},$$

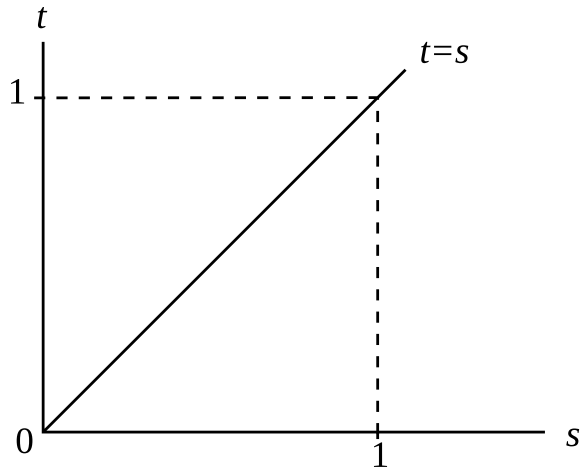
where $\mu > 0$ is a parameter.

(a) Using the fact that $\mathbb{E}(Y) = G'(1)$, show that $\mathbb{E}(Y) = \mu$. (2M)

(b) Let γ be the probability of eventual extinction. Working directly from the PGF of Y, find an expression for γ in terms of μ. Give your answer for the three cases (i) $\mu < 1$; (ii) $\mu = 1$; and (iii) $\mu > 1$. (5M)

(c) The diagram below shows a graph of $t = s$ for $0 \leq s \leq 1$. Make two copies of the diagram, one for the case $\mu < 1$, and the other for the case $\mu > 1$. For each copy, mark on it the following features:

(i) the curve $t = G(s)$;
(ii) the probability of eventual extinction, γ;
(iii) the mean, μ;
(iv) $\mathbb{P}(Y = 0)$. (4M)



(d) Now suppose that $\mu = 1$. Find $\mathbb{P}(Z_1 = 0)$ and $\mathbb{P}(Z_2 = 0)$. (2E)

(e) Continue to suppose that $\mu = 1$. Let $G_n(s)$ be the probability generating function of Z_n, for $n = 1, 2, \ldots$. Prove by mathematical induction that:

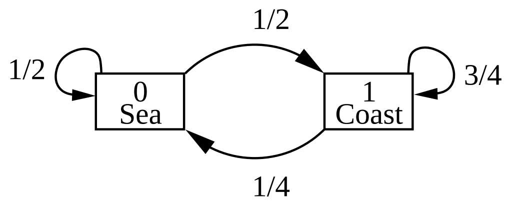
$$G_n(0) = \frac{n}{n+1} \quad \text{for } n = 1, 2, 3, \ldots. \qquad (\star)$$

You may assume that $G_{n+1}(s) = G\Big(G_n(s)\Big)$ for $n = 1, 2, \ldots$. Marks are awarded for setting out your answer clearly. (5M)

(f) Continue to suppose that $\mu = 1$. Using results from earlier parts of the question, find the probability that the population first goes extinct at generation $n = 10$. (2M)

[20 marks]

3. A certain type of whale spends its time either feeding out at sea (state 0), or resting in coastal waters (state 1). Each day it chooses one of these two states. Let $\{Y_1, Y_2, Y_3, \ldots\}$ be the state of the whale on days $1, 2, 3, \ldots$, where each $Y_t \in \{0, 1\}$. The whale's behaviour is modelled by a Markov chain following the transition diagram below.



(a) Write down the transition matrix, $\mathcal{P}$, for the Markov chain represented by this diagram. (2E)

(b) Suppose the whale is seen at the coast on day 1. Find the probability it will remain at the coast on day 2, and go out to sea on day 3. (2E)

(c) Show that $\boldsymbol{\pi}^T = \left(\frac{1}{3}, \frac{2}{3}\right)$ is an equilibrium distribution for the whale's Markov chain. (2E)

(d) Does the Markov chain converge to the equilibrium distribution $\boldsymbol{\pi}$ as $t \to \infty$? Explain why or why not. (3E)

Now suppose that the whole population consists of n whales, where n is a positive integer. Each individual whale has the same pattern of behaviour described by the Markov chain above, and all whales act independently of each other.

Let $Y_{1t}, Y_{2t}, \ldots, Y_{nt}$ denote the states of whales $1, 2, \ldots, n$ on day t. Thus:

$$Y_{it} = \begin{cases} 0 \text{ if whale } i \text{ is at sea on day } t, \\ 1 \text{ if whale } i \text{ is at the coast on day } t. \end{cases}$$

Let $\{X_1, X_2, X_3, \ldots\}$ be a Markov chain denoting the **total number of whales at the coast** on days $1, 2, 3, \ldots$. Thus:

$$X_t = \sum_{i=1}^{n} Y_{it} \quad \text{for} \quad t = 1, 2, 3, \ldots$$

(e) On average, how many whales are at the coast on day t, as $t \to \infty$? Justify your answer. (2M)

(f) Using your answers to previous parts of the question, write down the name of an equilibrium distribution for the Markov chain $\{X_1, X_2, X_3, \ldots\}$, and specify all parameters. Say whether the chain converges to this distribution as $t \to \infty$. (2M)

(g) Given X_t whales at the coast on day t, we can write X_{t+1} as the sum of two independent random variables:

$$X_{t+1} = U_{t+1} + W_{t+1}\,.$$

Here, U_{t+1} is the number of the X_t whales at the coast on day t that remain at the coast on day $t+1$. Give the appropriate definition of W_{t+1}, and name the distributions $[\,U_{t+1} \mid X_t\,]$ and $[\,W_{t+1} \mid X_t\,]$, specifying all parameters. (4M)

(h) Let Q be the transition matrix for the Markov chain $\{X_1, X_2, X_3, \ldots\}$, and let q_{rs} be the (r, s) element of Q. Which of choices A and B below is the correct formulation for q_{rs}? Give reasons for your answer.

A. $q_{rs} = \binom{n}{s}\left(\frac{2}{3}\right)^s\left(\frac{1}{3}\right)^{n-s}$

B. $q_{rs} = \sum_{u=0}^{r}\binom{r}{u}\left(\frac{3}{4}\right)^u\left(\frac{1}{4}\right)^{r-u}\binom{n-r}{s-u}\left(\frac{1}{2}\right)^{s-u}\left(\frac{1}{2}\right)^{n-r-s+u}$ (3H)

[20 marks]

4. Three Markov chains $\{X_0, X_1, X_2, \ldots\}$ are defined below by their transition matrices $\mathcal{P}$. The (i, j) element of $\mathcal{P}$ is written as p_{ij}.

A. Voter process or **Gene spread model:** $p_{ij} = \binom{N}{j} \left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{N-j}$ for $i, j \in \{0, 1, \ldots, N\}$.

B. Two-armed bandit process: $\mathcal{P} = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}$ where $\alpha, \beta \in (0, 1)$.

C. Random walk on the integers: $p_{ij} = \begin{cases} p & \text{if } j = i+1, \\ 1-p & \text{if } j = i-1, \\ 0 & \text{otherwise;} \end{cases}$ where $0 < p < 1$ and $i, j \in \mathbb{Z}$.

Three possible questions of interest about Markov chains are below:

Q1. Hitting probability for state 0. **Q2.** Expected time to absorption. **Q3.** Equilibrium distribution.

For each of processes A, B, and C, say whether each of questions Q1, Q2, and Q3 is of interest for the process. Briefly explain your answers.

[6 marks]

---

5. The Negative Binomial distribution with parameters k and p is defined to be the sum of k independent Geometric(p) random variables. That is, if $Y_i \sim$ Geometric(p) for all $i = 1, \ldots, k$, and $Y_1, \ldots, Y_k$ are independent, then $N = Y_1 + \ldots + Y_k$ has the Negative Binomial distribution with parameters k and p. We write $N \sim \text{NegBin}(k, p)$. Here, $k \in \mathbb{N}$, and $0 < p < 1$.

(a) Suppose $Y \sim$ Geometric(p), so $\mathbb{P}(Y = y) = p(1-p)^y$ for $y = 0, 1, 2, \ldots$ Working directly from the probability function of Y, show that the probability generating function (PGF) of Y is:

$$G_Y(t) = \mathbb{E}\left(t^Y\right) = \frac{p}{1-(1-p)t},$$

and state the range of values of t for which this expression is valid. (3E)

(b) Let $N \sim \text{NegBin}(k, p)$. Using the formulation $N = Y_1 + \ldots + Y_k$ where $Y_1, \ldots, Y_k$ are independent Geometric(p) random variables, show that the PGF of N is:

$$G_N(s) = \mathbb{E}\left(s^N\right) = \left\{\frac{p}{1-(1-p)s}\right\}^k,$$

and state the range of values of s for which this expression is valid. (2E)

(c) Now let X and Y be random variables defined as follows:

$$Y \sim \text{Geometric}\left(\tfrac{2}{5}\right), \qquad [\,X \,|\, Y\,] \sim \text{NegBin}\left(Y+1,\ \tfrac{2}{3}\right).$$

Show that the PGF of X is: $\quad G_X(s) = \mathbb{E}(s^X) = \dfrac{4}{9-5s},$

and hence name the distribution of X, with parameters. Show all your working. (6H)

(d) Let $Z = X + Y$. Show that $Z \sim \text{Geometric}(\frac{4}{15})$. Show all your working. (5H)

[16 marks]

6. Let $\{Z_0, Z_1, Z_2, \ldots\}$ be a branching process, where Z_n denotes the number of individuals born at time n, and $Z_0 = 1$. Let $Y \sim \text{Geometric}(p = 0.5)$ be the family size distribution.

Define $T^{(n)} = 1 + Z_1 + Z_2 + \ldots + Z_n$ to be the **total progeny up to generation n**, starting from a single ancestor, for $n = 1, 2, \ldots$. You may assume that $T^{(n)}$ satisfies the following recursive relationship:

$$T^{(n)} = 1 + T_1^{(n-1)} + T_2^{(n-1)} + \ldots + T_Y^{(n-1)} \quad \text{for } n = 2, 3, \ldots ,$$

where the random variables $T_1^{(n-1)}, T_2^{(n-1)}, \ldots, T_Y^{(n-1)}$ are independent of each other and of Y, and each have the same distribution as $T^{(n-1)}$.

Define T to be the **total progeny over all generations**. That is: $T = \lim_{n \to \infty} T^{(n)}$.

Define the following probability generating functions:

$$G(s) = \mathbb{E}\left(s^Y\right) ; \qquad H_n(s) = \mathbb{E}\left(s^{T^{(n)}}\right) \text{ for } n = 1, 2, \ldots ; \qquad H(s) = \mathbb{E}\left(s^T\right) .$$

You may assume that, for $Y \sim \text{Geometric}(p = 0.5)$, the probability generating function is

$$G(s) = \mathbb{E}\left(s^Y\right) = \frac{1}{2 - s} ,$$

and that all the PGFs listed above exist and are continuous for $-1 < s < 1$.

(a) Using the expression $T^{(n)} = 1 + T_1^{(n-1)} + T_2^{(n-1)} + \ldots + T_Y^{(n-1)}$ for random Y, show that

$$H_n(s) = \frac{s}{2 - H_{n-1}(s)} \quad \text{for } n = 2, 3, \ldots \qquad (\star)$$

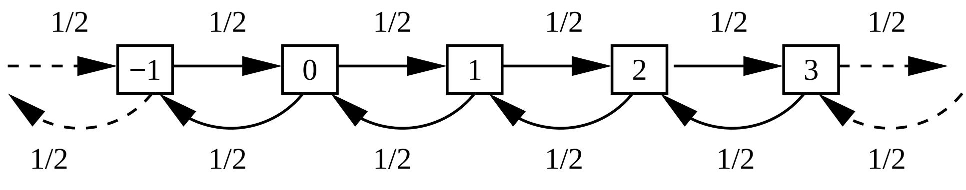
Show all your working. (6H)

(b) Using the expression $(\star)$, show that

$$H(s) = 1 - \sqrt{1 - s} .$$

Justify all steps in your working. You may assume that $H(s) = \lim_{n \to \infty} H_n(s)$ for $-1 < s < 1$. (4M)

Now suppose that $\{X_0, X_1, X_2, \ldots\}$ is a symmetric random walk on the integers, following the transition diagram below.



(c) Let W be the number of steps taken to **first return to state 0, starting at state 0.** For example, if $X_0 = 0$, $X_1 = 1$, and $X_2 = 0$, then $W = 2$ steps are taken to return from 0 to 0 again. Let $J(s) = \mathbb{E}\left(s^W\right)$ be the probability generating function of W. Show that

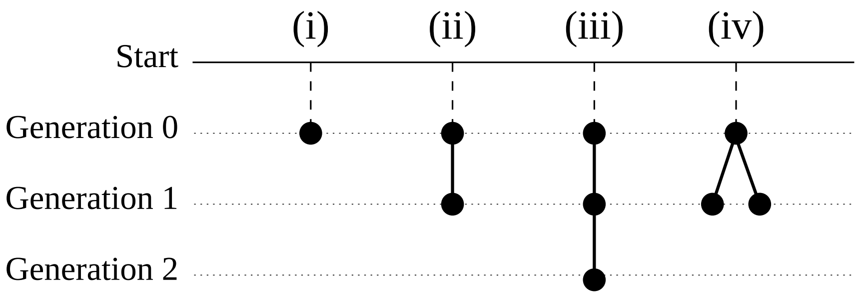
$$J(s) = 1 - \sqrt{1 - s^2} .$$

(6H)

(d) Using the PGFs shown in parts (b) and (c), specify the exact relationship between the random variables T and W. Hence explain why and how you might expect every **completed graph of a branching process** to correspond to a **path in a random walk.** Using the symbol A to indicate a step **away** from 0 in the random walk, and the symbol B to indicate a step **back** towards 0 in the random walk, suggest paths in the random walk that correspond to the four branching process graphs shown in (i), (ii), (iii), and (iv).

Note: You are *not* expected to prove that your suggested paths are correct. Marks are awarded for suggesting a suitable way of matching paths to graphs.

[**Hint:** the answer to (i) is AB.] (6H)



[22 marks]

ATTACHMENT

1. **Discrete Probability Distributions**

Distribution	$\mathbb{P}(X = x)$	$\mathbb{E}(X)$	$\text{Var}(X)$	PGF, $\mathbb{E}(s^X)$
Geometric(p)	pq^x (where $q = 1 - p$), for $x = 0, 1, 2, \ldots$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\frac{p}{1 - qs}$
	Number of failures before the first success in a sequence of independent trials, each with $\mathbb{P}(\text{success}) = p$.			
Binomial(n, p)	$\binom{n}{x} p^x q^{n-x}$ (where $q = 1 - p$), for $x = 0, 1, 2, \ldots, n$.	np	npq	$(ps + q)^n$
	Number of successes in n independent trials, each with $\mathbb{P}(\text{success}) = p$.			
Poisson(λ)	$\frac{\lambda^x}{x!} e^{-\lambda}$ for $x = 0, 1, 2, \ldots$	λ	λ	$e^{\lambda(s-1)}$

2. **Uniform Distribution:** $X \sim \text{Uniform}(a, b)$.
Probability density function, $f_X(x) = \frac{1}{b-a}$ for $a < x < b$. Mean, $\mathbb{E}(X) = \frac{a+b}{2}$.

3. **Properties of Probability Generating Functions**

Definition:	$G_X(s) = \mathbb{E}(s^X)$	
Moments:	$\mathbb{E}(X) = G'_X(1)$	$\mathbb{E}\Big\{ X(X-1)\ldots(X-k+1) \Big\} = G_X^{(k)}(1)$
Probabilities:	$\mathbb{P}(X = n) = \frac{1}{n!} G_X^{(n)}(0)$	

4. **Geometric Series:** $1 + r + r^2 + r^3 + \ldots = \sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$ for $|r| < 1$.

Finite sum: $\sum_{x=0}^{n} r^x = \frac{1 - r^{n+1}}{1 - r}$ for $r \neq 1$.

5. **Binomial Theorem:** For any $p, q \in \mathbb{R}$, and integer $n > 0$, $(p + q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

6. **Exponential Power Series:** For any $\lambda \in \mathbb{R}$, $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$.