Exam 2003 Solutions ①

STATS 325 Exam Solutions Semester 2, 2003

1a) A and B are independent $\Leftrightarrow$ $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Now $\mathbb{P}(A \cap \overline{B}) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$ by Partition Rule

$= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)$ by independence

$= \mathbb{P}(A)(1 - \mathbb{P}(B))$

$= \mathbb{P}(A)\mathbb{P}(\overline{B})$.

Thus A and $\overline{B}$ are independent, by definition.

b) Consider $A \cap B$: event $(A \cap B)$ is a subset of A and a subset of B, so

$\mathbb{P}(A \cap B) \leq \mathbb{P}(A) = \frac{4}{5}$

and $\mathbb{P}(A \cap B) \leq \mathbb{P}(B) = \frac{1}{2}$.

Only the smaller inequality is necessary, so $\mathbb{P}(A \cap B) \leq \frac{1}{2}$.

Now consider $A \cup B$:

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq 1$

$\Rightarrow \frac{4}{5} + \frac{1}{2} - \mathbb{P}(A \cap B) \leq 1$

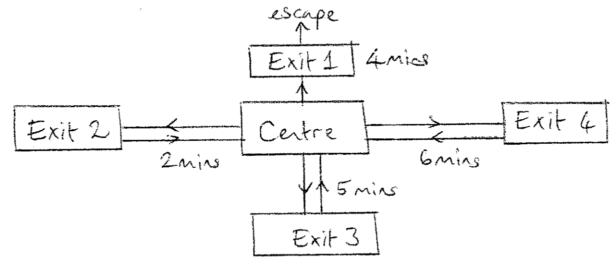
$\frac{4}{5} + \frac{1}{2} - 1 \leq \mathbb{P}(A \cap B)$

$\Rightarrow \frac{3}{10} \leq \mathbb{P}(A \cap B)$.

Thus overall we have $\frac{3}{10} \leq \mathbb{P}(A \cap B) \leq \frac{1}{2}$ as required.

②

2)



a) Using ideas of conditional expectation:

$\mathbb{E}(T) = \frac{1}{4}\{\mathbb{E}(T|\text{exit }1) + \dots + \mathbb{E}(T|\text{exit }4)\}$

$= \frac{1}{4}\{4 + (2 + \mathbb{E}T) + (5 + \mathbb{E}T) + (6 + \mathbb{E}T)\}$

$= \frac{1}{4} \cdot 17 + \frac{3}{4}\mathbb{E}(T)$

$\frac{1}{4}\mathbb{E}(T) = \frac{17}{4}$

$\Rightarrow \mathbb{E}(T) = 17$ mins.

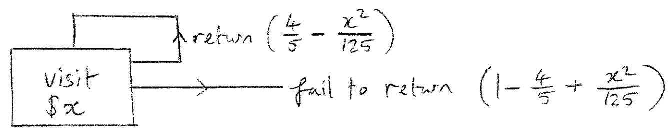
b) There are 3 exits that take time ≥ 3 minutes, all equally likely. Thus $\mathbb{P}(\text{Exit 1} \mid \text{time} \geq 3 \text{ mins}) = \frac{1}{3}$.

c) $\mathbb{P}(\text{escapes in 6 minutes}) = \mathbb{P}\left(\begin{array}{l}\text{chooses Exit 2 first,}\\ \text{then chooses Exit 1}\end{array}\right)$

because this is the only combination of exits that allow escape in 6 minutes.

Thus $\mathbb{P}(\text{escapes in 6 mins}) = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$.

3) a) Profit per visit $= \$x$.



$$\mathbb{E}P = x + \left(\frac{4}{5} - \frac{x^2}{125}\right)\mathbb{E}P$$

$$\Rightarrow \mathbb{E}P\left\{1 - \frac{4}{5} + \frac{x^2}{125}\right\} = x$$

$$\mathbb{E}P = \frac{x}{\frac{1}{5} + \frac{x^2}{125}}$$

$$\mathbb{E}P = \frac{125x}{25 + x^2}, \quad \text{as stated.}$$

b) Let $g(x) = \frac{125x}{25+x^2} = 125\,x\,(25+x^2)^{-1}$

Then the optimal x occurs where $g'(x) = 0$.

$$g'(x) = 125(25+x^2)^{-1} + 125x(-1)(25+x^2)^{-2}(2x) = 0$$

$$\Rightarrow \frac{125}{(25+x^2)^2}\left\{25 + x^2 - 2x^2\right\} = 0$$

$$\Rightarrow 25 + x^2 - 2x^2 = 0$$

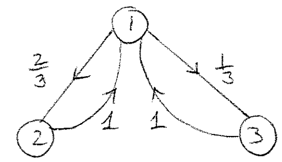
$$x^2 = 25$$

$$x = 5 \quad (\text{need } x>0).$$

Question states that any stationary point in $0 \le x \le 10$ is a maximum.

Thus the optimal profit per visit is $x = \$5$.

4a)



b) Equilibrium distribution, $\underset{\sim}{\pi}$, satisfies $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$ and $\pi_1 + \pi_2 + \pi_3 = 1$.

Thus
$$(\pi_1 \ \pi_2 \ \pi_3)\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (\pi_1 \ \pi_2 \ \pi_3)$$

$$\Rightarrow \begin{aligned} \pi_2 + \pi_3 &= \pi_1 & (1) \\ \tfrac{2}{3}\pi_1 &= \pi_2 & (2) \\ \tfrac{1}{3}\pi_1 &= \pi_3 & (3) \end{aligned}$$

Substituting for π_2, π_3 in eqn. $\pi_1 + \pi_2 + \pi_3 = 1$

$$\Rightarrow \pi_1 + \tfrac{2}{3}\pi_1 + \tfrac{1}{3}\pi_1 = 1$$

$$\Rightarrow \pi_1 = \tfrac{1}{2}.$$

Thus $\underset{\sim}{\pi} = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$.

c) The matrix P is irreducible, but all states have period 2 so it is not aperiodic.

Thus X_t does not converge to $\underset{\sim}{\pi}$ (or any other distribution) as $t \to \infty$.

5) a)

$$G_T(s) = \mathbb{E}(s^T)$$
$$= \mathbb{E}(s^{X_1+X_2+\dots+X_N})$$
$$= \mathbb{E}_N\{\mathbb{E}(s^{X_1+\dots+X_N} \mid N)\}$$
$$= \mathbb{E}_N\{\mathbb{E}(s^{X_1}s^{X_2}\dots s^{X_N} \mid N)\}$$
$$= \mathbb{E}_N\{\mathbb{E}(s^{X_1})\mathbb{E}(s^{X_2})\dots\mathbb{E}(s^{X_N})\}$$

because the X_i's are independent of each other and of N

$$= \mathbb{E}_N\{\mathbb{E}(s^{X_1})^N\}$$

because the X_i's are identically distributed

$$= \mathbb{E}_N\{G_X(s)^N\}$$

$$\underline{\underline{G_T(s) = G_N(G_X(s))}}$$ by definitions.

b) Z_n is the sum of offspring of the Z_{n-1} individuals at time $n-1$. These offspring can be labelled $Y_1, Y_2, \dots, Y_{Z_{n-1}}$.

Thus $Z_n = Y_1 + Y_2 + \dots + Y_{Z_{n-1}}$ (sum of a random number of random variables.)

Put
$$\left.\begin{array}{lll} T = Z_n & \Rightarrow & G_T = G_n \\ N = Z_{n-1} & \Rightarrow & G_N = G_{n-1} \\ X_i = Y_i & \Rightarrow & G_X = G \end{array}\right\}$$ in part (a).

Then $G_T(s) = G_N(G_X(s))$

$$\Rightarrow \underline{\underline{G_n(s) = G_{n-1}(G(s))}}$$ as required.

c) $\mathbb{P}(Z_n = 1) = G_n'(0)$ (general property of PGF : see Attachment).

Now $G_n'(s) = G_{n-1}'(G(s))\,G'(s)$

and $G(0) = \mathbb{P}(Y=0) = p_0$
$G'(0) = \mathbb{P}(Y=1) = p_1$

5c cont.) Thus

$$\mathbb{P}(Z_n = 1) = G_n'(0)$$
$$= G_{n-1}'(G(0))\,G'(0)$$
$$= G_{n-1}'(p_0)\,p_1$$
$$\underline{\underline{\mathbb{P}(Z_n=1) = p_1 G_{n-1}'(p_0)}}$$ as required.

Similarly, $\mathbb{P}(Z_n = 2) = \frac{1}{2!}G_n''(0)$

But $G_n''(s) = G_{n-1}''(G(s))\{G'(s)\}^2 + G_{n-1}'(G(s)).G''(s)$

and $G(0) = p_0$, $G'(0) = p_1$, $G''(0) = 2!\,p_2 = 2p_2$.

So
$$\mathbb{P}(Z_n=2) = \frac{1}{2!}\{G_{n-1}''(G(0))\{G'(0)\}^2 + G_{n-1}'(G(0)).G''(0)\}$$
$$= \frac{1}{2}\{G_{n-1}''(p_0)\,p_1^2 + G_{n-1}'(p_0).2p_2\}$$

$$\underline{\underline{\mathbb{P}(Z_n=2) = \frac{1}{2}p_1^2 G_{n-1}''(p_0) + p_2 G_{n-1}'(p_0).}}$$

6) a) $G(s) = \mathbb{E}(s^Y)$

$$= \sum_{y=0}^{\infty} s^y \, \mathbb{P}(Y=y)$$

$$= \sum_{y=0}^{\infty} s^y \left(\tfrac{1}{2}\right)^{y+1}$$

$$= \tfrac{1}{2} \cdot \sum_{y=0}^{\infty} \left(\tfrac{1}{2}s\right)^y$$

$$= \tfrac{1}{2} \cdot \frac{1}{1-\frac{1}{2}s} \quad \text{for } \left|\tfrac{1}{2}s\right| < 1, \text{ i.e. } -2<s<2$$

$$\underline{\underline{G(s) = \frac{1}{2-s}}} \quad \text{as stated.}$$

This is valid for $-2<s<2$, so $\underline{\underline{\text{radius of convergence} = 2}}$.

b) γ is the smallest solution ≥ 0 to $G(s)=s$:

$$G(s) = s \Rightarrow \frac{1}{2-s} = s \Rightarrow s^2 - 2s + 1 = 0$$
$$\Rightarrow (s-1)^2 = 0$$
$$\Rightarrow \underline{s=1}.$$

Thus $\underline{\underline{\gamma = \mathbb{P}(\text{extinction}) = 1}}$.

[Alternative: calculate $\mu = 1 \Rightarrow \gamma = 1$.]

c) i) $\mathbb{P}(Z_n = 0) = G_n(0)$

$$= \frac{n-(n-1)*0}{n+1 - n*0}$$

$$= \underline{\underline{\frac{n}{n+1}}} \quad \text{- formula correct for } r=0.$$

ii) $\mathbb{P}(Z_n = 1) = G_n'(0)$

Now $G_n(s) = \left(n-(n-1)s\right)\left(n+1-ns\right)^{-1}$

$$\Rightarrow G_n'(s) = -(n-1)(n+1-ns)^{-1} - (n-(n-1)s)(n+1-ns)^{-2}(-n)$$

6c cont.) So $G_n'(0) = \frac{-(n-1)}{n+1} + \frac{n^2}{(n+1)^2}$

$$= \frac{-(n+1)(n-1)}{(n+1)^2} + \frac{n^2}{(n+1)^2}$$

$$= \frac{-n^2+1+n^2}{(n+1)^2}$$

$$= \frac{1}{(n+1)^2}.$$

$$= \underline{\underline{\frac{n^0}{(n+1)^2}}} \quad \text{- formula correct for } r=1.$$

d) $\mathbb{P}(\text{extinct } \underline{\text{at}} \text{ generation } 8) = \mathbb{P}(Z_8 = 0) - \mathbb{P}(Z_7 = 0)$

$$= \frac{8}{9} - \frac{7}{8}$$

$$= \underline{\underline{\frac{1}{72} \quad (0.0139)}}.$$

e) $\mathbb{P}(Z_{10} \leq 20) = \mathbb{P}(Z_{10} = 0) + \sum_{r=1}^{20} \mathbb{P}(Z_{10} = r)$

$$= \frac{10}{11} + \sum_{r=1}^{20} \frac{10^{r-1}}{11^{r+1}} \quad \text{using formulas given}$$

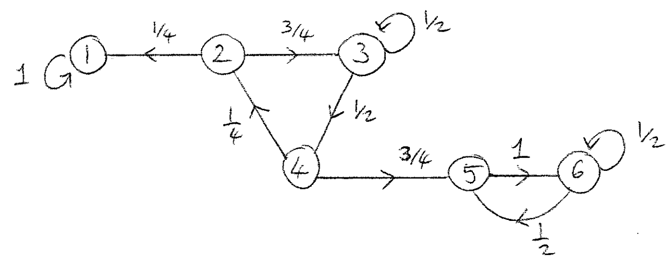
$$= \frac{10}{11} + \sum_{s=0}^{19} \frac{10^s}{11^{s+2}} \quad \text{putting } s = r-1$$

$$= \frac{10}{11} + \frac{1}{(11)^2} \sum_{s=0}^{19} \left(\frac{10}{11}\right)^s$$

$$= \frac{10}{11} + \frac{1}{121}\left(\frac{1-\left(\frac{10}{11}\right)^{20}}{1-\frac{10}{11}}\right) \quad \text{- finite sum of Geometric ser[ies] (in Attachment}$$

$$\underline{\underline{\mathbb{P}(Z_{10} \leq 20) = 0.986}}$$

7) a)



Communicating classes: $\{1\}$ closed
$\{2,3,4\}$ not closed
$\{5,6\}$ closed.

b) $\underset{\sim}{h}_A$ satisfies: $h_{1A} = 1$ because $A = \{1\}$

and $h_{iA} = \sum_{j=1}^{6} p_{ij} h_{jA}$ for $i \neq 1$. (*)

Now by inspection, $\underline{h_{5A} = h_{6A} = 0}$

Thus (*) $\Rightarrow$ $h_{2A} = \frac{1}{4} h_{1A} + \frac{3}{4} h_{3A}$

$\Rightarrow$ $h_{2A} = \frac{1}{4} + \frac{3}{4} h_{3A}$ (1)

and $h_{3A} = \frac{1}{2} h_{3A} + \frac{1}{2} h_{4A}$ $\Rightarrow$ $h_{3A} = h_{4A}$ (2)

and $h_{4A} = \frac{1}{4} h_{2A} + \frac{3}{4} \underset{0}{h_{5A}}$

$\Rightarrow$ $h_{4A} = \frac{1}{4} h_{2A}$ (3)

(2) in (1) $\Rightarrow$ $h_{2A} = \frac{1}{4} + \frac{3}{4} h_{4A}$

using (3) $\Rightarrow$ $h_{2A} = \frac{1}{4} + \frac{3}{4}\left(\frac{1}{4} h_{2A}\right)$

$\frac{13}{16} h_{2A} = \frac{1}{4}$ $\Rightarrow$ $\underline{h_{2A} = \frac{4}{13}}$.

So also, $h_{3A} = h_{4A} = \frac{1}{4} h_{2A} \underline{= \frac{1}{13}}$.

Thus $\underline{\underline{\underset{\sim}{h}_A = \left(1, \frac{4}{13}, \frac{1}{13}, \frac{1}{13}, 0, 0\right)}}$.

8) a) If $X_0 \sim (0.1, 0.9)^T$, then

$$X_1 \sim (0.1, 0.9)\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} = \big(0.1(1-\alpha) + 0.9\beta,\ 0.1\alpha + 0.9(1-\beta)\big)$$

So $\underline{\underline{X_1 \sim \big(0.1(1-\alpha) + 0.9\beta,\quad 0.1\alpha + 0.9(1-\beta)\big)}}$

b) $\mathbb{P}(X_1 = 2, X_2 = 1, X_3 = 2 \mid X_0 = 2) = p_{22}\, p_{21}\, p_{12}$

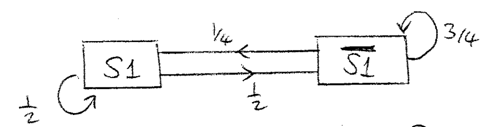
$\underline{\underline{= (1-\beta)\beta\alpha}}$.

c) Y_t can be in one of two states:
state 1 or not state 1.
Call these S_1 and $\overline{S_1}$.

Now $\mathbb{P}(Y_{t+1} = 1 \mid Y_t = 1) = \frac{1}{2}$ from matrix given (Q_{11}).

Also, $\mathbb{P}(Y_{t+1} = 1 \mid Y_t = 2) = \frac{1}{4} = Q_{21}$
$\mathbb{P}(Y_{t+1} = 1 \mid Y_t = 3) = \frac{1}{4} = Q_{31}$

So whenever Y_t is in state $\overline{S_1}$ (i.e. state 2 or 3),
$\mathbb{P}(Y_{t+1} = S_1 \mid Y_t = \overline{S_1}) = \frac{1}{4}$.



The 2-state matrix is therefore $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$ i.e. $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$

So $\mathbb{P}(Y_t = 1 \mid Y_0 = 1) = \mathbb{P}(Y_t = S_1 \mid Y_0 = S_1) = (P^t)_{11}$

$= \frac{1}{\alpha+\beta}\left\{\beta + \alpha(1-\alpha-\beta)^t\right\}$

$= \frac{1}{\frac{1}{2}+\frac{1}{4}}\left\{\frac{1}{4} + \frac{1}{2}\left(\frac{1}{4}\right)^t\right\}$

$\underline{\underline{\mathbb{P}(Y_t = 1 \mid Y_0 = 1) = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{4}\right)^t}}$ general solution

9) The transition matrix has components $\mathbb{P}(Z_{t+1} = j \mid Z_t = i)$.

Suppose $Z_t = 0$: then $\mathbb{P}(Z_{t+1} = 0 \mid Z_t = 0) = 1$ (given in Q).

Suppose $Z_t = 1$: then 1 individual has $\begin{cases} 0 \text{ offspring} & (\text{prob } 0) \\ 1 \quad " & (\text{prob } 1). \end{cases}$

Suppose $Z_t = 2$: 2 indivs each have $\begin{cases} 0 \text{ young} & (\text{prob. } 1-\frac{1}{2} = \frac{1}{2}) \\ 1 \text{ young} & (\text{prob. } \frac{1}{2}) \end{cases}$

$$\Rightarrow Z_{t+1} = \begin{cases} 0 & \text{prob. } (\frac{1}{2})^2 = \frac{1}{4} \\ 1 & \text{prob. } 2 * \frac{1}{2} * \frac{1}{2} = \frac{1}{2} \\ 2 & \text{prob. } (\frac{1}{2})^2 = \frac{1}{4}. \end{cases}$$

Suppose $Z_t = 3$: 3 indivs each have $\begin{cases} 0 \text{ young} & (\text{prob } 1-\frac{1}{3} = \frac{2}{3}) \\ 1 \text{ young} & (\text{prob } \frac{1}{3}). \end{cases}$

$$\Rightarrow Z_{t+1} = \begin{cases} 0 & \text{prob } (\frac{2}{3})^3 = \frac{8}{27} \\ 1 & \text{prob } \binom{3}{1}\frac{1}{3}(\frac{2}{3})^2 = \frac{12}{27} \\ 2 & \text{prob } \binom{3}{2}(\frac{1}{3})^2\frac{2}{3} = \frac{6}{27} \\ 3 & \text{prob } (\frac{1}{3})^3 = \frac{1}{27} \end{cases} \quad \text{Binomial}(3, \tfrac{1}{3})$$

The transition matrix is :

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{8}{27} & \frac{12}{27} & \frac{6}{27} & \frac{1}{27} \end{pmatrix} \end{array}.$$

$\mathbb{P}(\text{extinction}) = h_{30} = \mathbb{P}(\text{hit state 0 starting at state 3})$.

Now the hitting probabilities $\{h_{i0}\}$ satisfy:

$$h_{00} = 1$$
$$h_{10} = 0 \quad \text{by inspection}$$
$$h_{20} = \tfrac{1}{4} h_{00} + \tfrac{1}{2} h_{10} + \tfrac{1}{4} h_{20}$$
$$\Rightarrow \tfrac{3}{4} h_{20} = \tfrac{1}{4} + 0 \Rightarrow \underline{h_{20} = \tfrac{1}{3}}$$

and $h_{30} = \frac{8}{27} h_{00} + \frac{12}{27} h_{10} + \frac{6}{27} h_{20} + \frac{1}{27} h_{30}$

$$\Rightarrow \tfrac{26}{27} h_{30} = \tfrac{8}{27} + 0 + \tfrac{6}{27} * \tfrac{1}{3} = \tfrac{10}{27}$$

$$\underline{\underline{\mathbb{P}(\text{extinction}) = h_{30} = \tfrac{5}{13} \quad (0.385).}}$$

10) $G_k(s) = \mathbb{E}(s^{T_k})$.

Now $T_1 = 0$ with probability 1, so $G_1(s) = \mathbb{E}(s^{T_1}) = 1$.

Also $T_2 = \begin{cases} 1 & \text{with prob. } 1/4 \\ 1 + T_3 & \text{with prob. } 3/4 \end{cases}$

So $G_2(s) = \mathbb{E}(s^{T_2}) = \frac{1}{4}\mathbb{E}(s^1) + \frac{3}{4}\mathbb{E}(s^{1+T_3})$

$$\therefore \underline{G_2(s) = \tfrac{1}{4} s\,(1 + 3G_3(s)).} \quad \circledast$$

Also $T_3 = \begin{cases} 1 + T_2 & \text{with prob. } 1/4 \\ 1 + T_3 & \text{with prob. } 3/4 \end{cases}$

So $G_3(s) = \mathbb{E}(s^{T_3}) = \frac{1}{4}\mathbb{E}(s^{1+T_2}) + \frac{3}{4}\mathbb{E}(s^{1+T_3})$

$$= \tfrac{1}{4} s\,(G_2(s) + 3G_3(s))$$
$$= \tfrac{1}{4} s \left\{ \tfrac{1}{4} s (1 + 3G_3(s)) + 3G_3(s) \right\} \quad \text{from } \circledast$$
$$= \tfrac{1}{16} s^2 + \tfrac{3}{16} s^2 G_3(s) + \tfrac{3}{4} s\, G_3(s)$$

$$G_3(s) \left\{ 1 - \tfrac{3}{16} s^2 - \tfrac{3}{4} s \right\} = \tfrac{1}{16} s^2$$

$$G_3(s) = \frac{s^2}{16\left(1 - \frac{3}{16}s^2 - \frac{3}{4}s\right)}$$

$$\underline{\underline{G_3(s) = \frac{s^2}{16 - 12s - 3s^2}}}$$