Exam 2004 Solutions

STATS 325 Exam Solutions Semester 2, 2004

1a) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \mid B)\,\mathbb{P}(B)$

$= \{1 - \mathbb{P}(\bar{A})\} + \mathbb{P}(B)\{1 - \mathbb{P}(A \mid B)\}$

$= 1 - \mathbb{P}(\bar{A}) + \mathbb{P}(B) * \mathbb{P}(\bar{A} \mid B)$

$\underline{\underline{\mathbb{P}(A \cup B) = 1 - \mathbb{P}(\bar{A}) + \mathbb{P}(\bar{A} \cap B)}}$ as stated.

b) $\mathbb{P}(A \cup B) = 1 - \mathbb{P}(\bar{A}) + \mathbb{P}(\bar{A} \cap B)$ from (a)

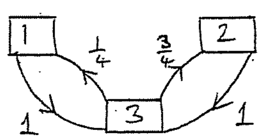
$= 1 - \mathbb{P}(\bar{A}) + \mathbb{P}(B \mid \bar{A})\,\mathbb{P}(\bar{A})$

$= 1 - \mathbb{P}(\bar{A})\{1 - \mathbb{P}(B \mid \bar{A})\}$

$= 1 - \mathbb{P}(\bar{A}) * \mathbb{P}(\bar{B} \mid \bar{A})$

$\underline{\underline{\mathbb{P}(A \cup B) = 1 - \mathbb{P}(\bar{A} \cap \bar{B})}}$. as stated.

2a)



b) Equilibrium distribution, $\underset{\sim}{\pi}$, satisfies:

$$\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T \Rightarrow (\pi_1\ \pi_2\ \pi_3)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix} = (\pi_1\ \pi_2\ \pi_3)$$

$$\Rightarrow \quad \frac{1}{4}\pi_3 = \pi_1 \quad (1)$$
$$\frac{3}{4}\pi_3 = \pi_2 \quad (2)$$
$$\pi_1 + \pi_2 = \pi_3 \quad (3)$$

2b cont.) Also, $\pi_1 + \pi_2 + \pi_3 = 1$. (4)

(3) in (4) $\Rightarrow$ $2\pi_3 = 1 \Rightarrow \underline{\underline{\pi_3 = \frac{1}{2}}}$.

Thus (1) $\Rightarrow$ $\pi_1 = \frac{1}{4}\pi_3 = \frac{1}{4} * \frac{1}{2} = \underline{\frac{1}{8}}$

(2) $\Rightarrow$ $\pi_2 = \frac{3}{4}\pi_3 = \frac{3}{4} * \frac{1}{2} = \underline{\frac{3}{8}}$

So equilibrium distribution is $\underline{\underline{\underset{\sim}{\pi} = \left(\frac{1}{8}, \frac{3}{8}, \frac{1}{2}\right)}}$.

c) The chain is irreducible, but it is <u>NOT</u> aperiodic, because all states have period 2.

Thus $\{X_t\}_{t \geq 0}$ does <u>NOT</u> converge to $\underset{\sim}{\pi}$ or any other distribution as $t \to \infty$.

3)a) $a = \gamma \quad b = 1 \quad c = 1 \quad d = \gamma \quad e = \mathbb{P}(Y = 0)$.

b) μ is the value of the gradient at point $s = 1$: that is, $\mu = \mathbb{E}(Y) = G'(1)$.

Because the slope is steeper than the line $t = s$ at point $s = 1$, it follows that $\underline{\underline{\mu > 1}}$.

Alternative explanation: $\mu > 1$ because $\gamma < 1$. If $\mu \leq 1$ then extinction is certain and $\gamma = 1$.

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4) a) If $X_0 \sim \underline{\pi}^T$, then $X_1 \sim \pi^T P$. So the distribution of X_1 is:

$$X_1 \sim (0.4 \;\; 0.6) \begin{pmatrix} 3/5 & 2/5 \\ 1/4 & 3/4 \end{pmatrix} = (0.39 \;\; 0.61)$$

Thus $\underline{X_1 \sim (0.39 \;\; 0.61)^T}$

b) $\mathbb{P}(X_0 = 2, X_2 = 2, X_4 = 1) = \mathbb{P}(X_0 = 2) * (P^2)_{22} * (P^2)_{21}$ (*)

Now $P^2 = \begin{pmatrix} 3/5 & 2/5 \\ 1/4 & 3/4 \end{pmatrix}\begin{pmatrix} 3/5 & 2/5 \\ 1/4 & 3/4 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ 27/80 & 53/80 \end{pmatrix}$

(Only 2nd row of P^2 is needed in (*).)

So $\mathbb{P}(X_0 = 2, X_2 = 2, X_4 = 1) = 0.6 * \left(\frac{53}{80}\right) * \left(\frac{27}{80}\right)$

$= \underline{0.134}$.

c) As $t \to \infty$, $P^t \to \begin{pmatrix} 5/13 & 8/13 \\ 5/13 & 8/13 \end{pmatrix}$ because $\left(\frac{7}{20}\right)^t \to 0$.

Thus for ANY starting state i, where $i=1$ or $i=2$,

$\mathbb{P}(X_t = 1 \mid X_0 = i) \to \frac{5}{13}$ (i.e. $(P^t)_{i1}$)

$\mathbb{P}(X_t = 2 \mid X_0 = i) \to \frac{8}{13}$ (i.e. $(P^t)_{i2}$).

Thus $\{X_t\}$ converges to a fixed distribution as $t \to \infty$, where the distribution is independent of start state i and independent of time, t.

Thus $\{X_t\}$ DOES converge to an equilibrium distribution, and the distribution is $\underline{\left(\frac{5}{13}, \frac{8}{13}\right)}$.

5) a) $G(s) = \mathbb{E}(s^Y) = \sum_{y=0}^{\infty} s^y \, \mathbb{P}(Y=y)$

$= \sum_{y=0}^{\infty} s^y \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^y$

$= \frac{1}{4} \sum_{y=0}^{\infty} \left(\frac{3s}{4}\right)^y$

$= \frac{1}{4} \cdot \frac{1}{1 - \frac{3s}{4}}$ for $\left|\frac{3s}{4}\right| < 1$.

$\therefore \; G(s) = \frac{1}{4-3s}$ for $|s| < \frac{4}{3}$, as stated.

b) $G_2(s) = G(G(s)) = \frac{1}{4 - 3G(s)}$

$= \frac{1}{4 - 3\left(\frac{1}{4-3s}\right)}$

$= \frac{4-3s}{4(4-3s) - 3}$

$\underline{G_2(s) = \frac{4-3s}{13-12s}}$

c) The probability of eventual extinction, γ, is the smallest solution ≥ 0 to the equation $G(s) = s$.

Thus $G(s) = s \implies \frac{1}{4-3s} = s$

$\implies 4s - 3s^2 = 1$

$\implies 3s^2 - 4s + 1 = 0$

$(3s-1)(s-1) = 0$

$\implies \underline{s = \frac{1}{3} \text{ or } s = 1}$.

γ is the smallest solution ≥ 0, so $\underline{\gamma = \frac{1}{3}}$.

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5d) $\mathbb{P}(\text{extinct } \underline{\text{at}} \text{ generation } n=3) = G_3(0) - G_2(0)$

Now $G_3(0) = G_2(G(0))$

$$= \frac{4 - 3G(0)}{13 - 12G(0)} \quad \text{from (b)}$$

and $G(0) = \frac{1}{4 - 3*0} = \frac{1}{4}$.

So $G_3(0) = \frac{4 - 3/4}{13 - 12/4} = \underline{\frac{13}{40}}$

Also, $G_2(0) = \frac{4-0}{13-0} = \underline{\frac{4}{13}}$

So $\mathbb{P}(\text{extinct } \underline{\text{at}} \text{ generation } n=3) = \frac{13}{40} - \frac{4}{13}$

$$= \frac{9}{520} \quad (0.0173).$$

e) $G(s) = \mathbb{E}(s^Y) = \sum_{y=0}^{\infty} s^y . \mathbb{P}(Y=y)$

$$= \sum_{y=0}^{\infty} \frac{(0.5s)^y}{y!} e^{-0.5}$$

$$= e^{-0.5} . e^{0.5s}$$

$\underline{G(s) = e^{0.5(s-1)}}$ as stated. Valid for $s \in \mathbb{R}$.

f) If $Y \sim \text{Poisson}(\lambda = 0.5)$, then $\mathbb{E}(Y) = 0.5$.

Thus $\mu = \mathbb{E}(Y) < 1$, so $\underline{\gamma = 1.}$

(Extinction is definite if $\mu < 1$.)

6a) Consider $T = \begin{cases} 1 & \text{with probability } 3/4 \\ 1 + T_1 + T_2 & \text{w.p. } 1/4, \text{ where } T_1 \sim T, \; T_2 \sim T \end{cases}$

Then $H(s) = \mathbb{E}(s^T) = \frac{3}{4} s^1 + \frac{1}{4} \mathbb{E}(s^{1+T_1+T_2})$

$$= \frac{3}{4} s + \frac{1}{4} s \, \mathbb{E}(s^{T_1+T_2}).$$

where T_1 represents time to go from state -1 to 0,
T_2 " " " " " " 0 to 1,

and T_1, T_2 are independent and identically distributed, with PGF $H(s)$.

So $H(s) = \frac{3}{4} s + \frac{1}{4} s \, \mathbb{E}(s^{T_1}) \, \mathbb{E}(s^{T_2})$ by independence

$$= \frac{3}{4} s + \frac{1}{4} s \{H(s)\}^2$$

$\Rightarrow \quad s\{H(s)\}^2 - 4H(s) + 3s = 0$

$\Rightarrow \quad H(s) = \frac{4 \pm \sqrt{16 - 4*3s*s}}{2s}$

$\Rightarrow \quad H(s) = \frac{4 \pm \sqrt{16 - 12s^2}}{2s}$ as stated.

b) $H(s) = \frac{2*2 \pm \sqrt{4(4 - 3s^2)}}{2s}$

$$= \frac{2*2 \pm 2\sqrt{4 - 3s^2}}{2s}$$

$H(s) = \frac{2 \pm \sqrt{4 - 3s^2}}{s}$ as stated.

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6c) $H(0) = \mathbb{P}(T=0) = 0$, because it must take at least one step to get from state 0 to state 1.

Under the (+) root, $H(0) = \dfrac{2+\sqrt{4}}{0} = \infty$.

So it can not be the (+) root that is correct.

d) T is defective if and only if $H(1) < 1$.

Now $H(1) = \dfrac{2-\sqrt{4-3}}{1} = \dfrac{2-1}{1} = 1$.

So $\underline{\underline{T \text{ is not defective.}}}$

e) $\mathbb{E}(T) = H'(1)$.

Now $H(s) = 2s^{-1} - \sqrt{\dfrac{4}{s^2} - \dfrac{3s^2}{s^2}}$

$= 2s^{-1} - (4s^{-2}-3)^{1/2}$

So $H'(s) = -2s^{-2} - \frac{1}{2}(4s^{-2}-3)^{-1/2} * (-2)(4s^{-3})$

$\Rightarrow H'(1) = -2 - \frac{1}{2}(4-3)^{-1/2} . * (-8)$

$= -2 + 4$

$= 2$

So $\underline{\underline{\mathbb{E}(T) = H'(1) = 2 \text{ steps.}}}$

f) $W = T_1 + T_2 + T_3 + T_4$ where $T_1, \dots, T_4$ are indept, identically distributed, with common PGF $H(s)$.

[T_i = time taken to reach state i from state $i-1$.]

So $G(s) = \mathbb{E}(s^W) = \mathbb{E}(s^{T_1}) \dots \mathbb{E}(s^{T_4})$ by independence

$\therefore \underline{\underline{G(s) = \{H(s)\}^4}}$.

7) a)

$$\begin{tikzcd}
\boxed{0} \arrow[r, "1"] & \boxed{1} \arrow[r, "\frac{5}{6}"] \arrow[loop below, "\frac{1}{6}"] & \boxed{2} \arrow[r, "\frac{4}{6}"] \arrow[loop below, "\frac{2}{6}"] & \boxed{3} \arrow[r, "\frac{3}{6}"] \arrow[loop below, "\frac{3}{6}"] & \boxed{4} \arrow[r, "\frac{2}{6}"] \arrow[loop below, "\frac{4}{6}"] & \boxed{5} \arrow[r, "\frac{1}{6}"] \arrow[loop below, "\frac{5}{6}"] & \boxed{6} \arrow[loop below, "1"]
\end{tikzcd}$$

b) Let $(m_0, m_1, \dots, m_6)$ be the vector of expected hitting times from states $0, 1, \dots, 6$ to state 6.

Then $\mathbb{E}(T) = m_0$.

Now $m_0 = 1 + m_1$

$m_1 = 1 + \frac{1}{6}m_1 + \frac{5}{6}m_2 \Rightarrow \frac{5}{6}(m_1 - m_2) = 1$
$\Rightarrow \underline{m_1 - m_2 = \frac{6}{5}}$.

$m_2 = 1 + \frac{2}{6}m_2 + \frac{4}{6}m_3 \Rightarrow \frac{4}{6}(m_2 - m_3) = 1$
$\Rightarrow \underline{m_2 - m_3 = \frac{6}{4}}$.

Similarly, $m_3 = 1 + \frac{3}{6}m_3 + \frac{3}{6}m_4 \Rightarrow \underline{m_3 - m_4 = \frac{6}{3}}$

$m_4 = 1 + \frac{4}{6}m_4 + \frac{2}{6}m_5 \Rightarrow \underline{m_4 - m_5 = \frac{6}{2}}$

$m_5 = 1 + \frac{5}{6}m_5 + \frac{1}{6}m_6 \Rightarrow \underline{m_5 - m_6 = 6}$

and $\underline{m_6 = 0 \text{ by inspection.}}$

Thus $m_5 = 0 + 6 = 6$

$m_4 = m_5 + \frac{6}{2} = 6 + 3 = 9$

$m_3 = m_4 + \frac{6}{3} = 9 + 2 = 11$

$m_2 = m_3 + \frac{6}{4} = 11 + \frac{6}{4} = 12.5$

$m_1 = m_2 + \frac{6}{5} = 12.5 + \frac{6}{5} = 13.7$

$\therefore \underline{\underline{m_0 = \mathbb{E}(T) = 1 + m_1 = 14.7 \text{ tosses.}}}$

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8) a) When $k=0$: $LHS = h_0 = 1$
$RHS = 0 * h_1 - (0-1) = 1 = LHS$.

So formula ⊛ holds for $k=0$.

When $k=1$: $LHS = h_1$
$RHS = 1 * h_1 - (1-1) = h_1 = LHS$.

So formula ⊛ holds for $k=1$.

b) Consider $h_r = \frac{1}{2} h_{r-1} + \frac{1}{2} h_{r+1}$ for $r = 1, 2, 3, \ldots$

$\Rightarrow \quad 2h_r = h_{r-1} + h_{r+1}$

$\Rightarrow \quad h_{r+1} = 2h_r - h_{r-1}$ for $r = 1, 2, 3, \ldots$ as required.

c) Suppose ⊛ holds for all $k = 0, 1, \ldots, r$.

Then $h_{r+1} = 2h_r - h_{r-1}$ by (b)

$= 2(rh_1 - (r-1)) - ((r-1)h_1 - (r-2))$

because ⊛ holds for $k=r$ and $k=r-1$

$= h_1(2r - r + 1) - 2r + 2 + r - 2$

$= h_1(r+1) - r$

$= kh_1 - (k-1)$ where $k = r+1$.

Thus ⊛ is true for $k = r+1$, so inductive proof from (a) and (c) proves ⊛ for all $k = 0, 1, 2, \ldots$

d) We have $h_k = kh_1 - (k-1)$, so $0 \leqslant kh_1 - (k-1) \leqslant 1 \quad \forall k$.

So (rearranging), $\frac{k-1}{k} \leqslant h_1 \leqslant \frac{k}{k} = 1 \quad \forall k = 1, 2, \ldots$

Let $k \to \infty$, then $\frac{k-1}{k} \to 1$. It follows that $h_1 = 1$.

9) a) $\mathbb{E}(T) = 1 + \frac{1}{2}\mathbb{E}(T) + \frac{1}{2}\mathbb{E}(U)$.

$\Rightarrow \quad \frac{1}{2}\mathbb{E}(T) = 1 + \frac{1}{2}\mathbb{E}(U)$. (1)

But $\mathbb{E}(U) = 1 + \frac{1}{2}\mathbb{E}(U) + \frac{1}{2} * 0$

$\Rightarrow \quad \frac{1}{2}\mathbb{E}(U) = 1$, so $\mathbb{E}(U) = 2$ as stated.

Substitute in (1): $\mathbb{E}(T) = 2 + \mathbb{E}(U)$
$= 2 + 2$
$\therefore \quad \mathbb{E}(T) = 4$.

b) Consider $U = \begin{cases} 1 & \text{with probability } \frac{1}{2} \quad \text{[go to state 3]} \\ 1 + U_1 & \text{w.p. } \frac{1}{2}, \text{ where } U_1 \sim U. \quad \text{[stay in state 2]} \end{cases}$

Then $U^2 = \begin{cases} 1^2 = 1 & \text{w.p. } \frac{1}{2} \\ (1+U_1)^2 = 1 + 2U_1 + U_1^2 & \text{w.p. } \frac{1}{2}. \end{cases}$

So $\mathbb{E}(U^2) = \frac{1}{2} * 1 + \frac{1}{2} * (1 + 2\mathbb{E}(U_1) + \mathbb{E}(U_1^2))$

But $\mathbb{E}(U_1) = \mathbb{E}(U) = 2$ (from part (a)), because $U_1 \sim U$.
And $\mathbb{E}(U_1^2) = \mathbb{E}(U^2)$.

So $\mathbb{E}(U^2) = \frac{1}{2} + \frac{1}{2}(1 + 2*2 + \mathbb{E}(U^2))$

$= 3 + \frac{1}{2}\mathbb{E}(U^2)$

$\Rightarrow \quad \frac{1}{2}\mathbb{E}(U^2) = 3$

$\Rightarrow \quad \mathbb{E}(U^2) = 6$.

Now consider $T = \begin{cases} 1 + U & \text{w.p. } \frac{1}{2} \quad \text{[go to state 2]} \\ 1 + T_1 & \text{w.p. } \frac{1}{2}, \quad \text{[stay in state 1]} \end{cases}$

where $T_1 \sim T$.

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9b cont.) Thus $T^2 = \begin{cases} (1+U)^2 & \text{w.p. } \frac{1}{2} \\ (1+T_1)^2 & \text{w.p. } \frac{1}{2} \end{cases}$

i.e. $T^2 = \begin{cases} 1 + 2U + U^2 & \text{w.p. } \frac{1}{2} \\ 1 + 2T_1 + T_1^2 & \text{w.p. } \frac{1}{2} \end{cases}$

from which,

$$\mathbb{E}(T^2) = \tfrac{1}{2}\left(1 + 2\mathbb{E}(U) + \mathbb{E}(U^2)\right) + \tfrac{1}{2}\left(1 + 2\mathbb{E}(T_1) + \mathbb{E}(T_1^2)\right)$$

Now $T_1 \sim T$, so $\mathbb{E}(T_1) = \mathbb{E}(T) = 4$ (from part (a))
and $\mathbb{E}(T_1^2) = \mathbb{E}(T^2)$.

Also, $\mathbb{E}U = 2$ from (a) and $\mathbb{E}(U^2) = 6$ from above.

So $\mathbb{E}(T^2) = \frac{1}{2}(1 + 2*2 + 6) + \frac{1}{2}(1 + 2*4 + \mathbb{E}(T^2))$

$\frac{1}{2}\mathbb{E}(T^2) = 10$

$\therefore$ $\underline{\underline{\mathbb{E}(T^2) = 20}}$

Hence $Var(T) = \mathbb{E}(T^2) - (\mathbb{E}T)^2$

$= 20 - 4^2$

$\underline{\underline{Var(T) = 4}}$