STATS 325 — Exam Solutions — Semester 2, 2005

1a) $LHS = \mathbb{P}(A \cup B)$

$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$= \{ \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \bar{B}) \} + \{ \cancel{\mathbb{P}(B \cap A)} + \mathbb{P}(B \cap \bar{A}) \} - \cancel{\mathbb{P}(A \cap B)}$

(Partition Thm for $\mathbb{P}(A)$ and $\mathbb{P}(B)$)

$= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap \bar{B}) + \mathbb{P}(\bar{A} \cap B)$

$= RHS.$

Note: any correct solution is acceptable.
It is also acceptable to prove $RHS = LHS$.

1b) $LHS = RHS$ = probability of either A or B, but not both.

1c) Consider $\mathbb{P}(A \cap B) = \underbrace{\mathbb{P}(B|A)}_{\leq 1} \mathbb{P}(A) \leq \mathbb{P}(A)$

So $\mathbb{P}(A \cap B) \leq \frac{1}{3}$ (*)

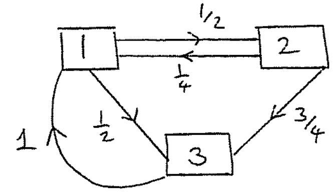
Similarly, $\mathbb{P}(A \cap B) \leq \mathbb{P}(B) = \frac{3}{4}$ (redundant by (*)).

Now consider $\mathbb{P}(A \cup B) \leq 1$

$$\Rightarrow \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq 1$$
$$\frac{1}{3} + \frac{3}{4} - \mathbb{P}(A \cap B) \leq 1$$
$$\frac{1}{3} + \frac{3}{4} - 1 \leq \mathbb{P}(A \cap B)$$
$$\therefore \quad \frac{1}{12} \leq \mathbb{P}(A \cap B). \quad (**)$$

Overall: $\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}$ by (*), (**).

2a)



b) Require $\underset{\sim}{\pi}$ such that $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$ and $\pi_1 + \pi_2 + \pi_3 = 1$.

Now $$\underset{\sim}{\pi}^T P = (\pi_1 \;\; \pi_2 \;\; \pi_3) \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\pi_2 + \pi_3 \\ \frac{1}{2}\pi_1 \\ \frac{1}{2}\pi_1 + \frac{3}{4}\pi_2 \end{pmatrix}$$

Equations required:
$$\frac{1}{4}\pi_2 + \pi_3 = \pi_1 \quad (1)$$
$$\frac{1}{2}\pi_1 = \pi_2 \quad (2)$$
$$\frac{1}{2}\pi_1 + \frac{3}{4}\pi_2 = \pi_3 \quad (3)$$

Also
$$\pi_1 + \pi_2 + \pi_3 = 1 \quad (4)$$

(2) in (1) for π_1: $\frac{1}{4}\pi_2 + \pi_3 = 2\pi_2$

$\Rightarrow \pi_3 = \frac{7}{4}\pi_2$ (5)

(5) and (2) into (4): $2\pi_2 + \pi_2 + \frac{7}{4}\pi_2 = 1$

$\frac{19}{4}\pi_2 = 1$

$\pi_2 = \frac{4}{19}$. (6)

Thus (2), (5) and (6) give $\underset{\sim}{\pi} = \left(\frac{8}{19}, \frac{4}{19}, \frac{7}{19} \right)$

2c) The Markov chain is irreducible and aperiodic, and an equilibrium distribution exists.
So the chain X_t does converge to $\underset{\sim}{\pi}$ as $t \to \infty$.

3) a) $G(s) = \mathbb{E}(s^Y)$

$$= \sum_{y=0}^{n} s^y \, \mathbb{P}(Y=y)$$

$$= \sum_{y=0}^{n} s^y \cdot \binom{n}{y} p^y q^{n-y}$$

$$= \sum_{y=0}^{n} \binom{n}{y} (ps)^y q^{n-y}$$

$$= \underline{\underline{(ps+q)^n}} \text{ by the Binomial Theorem.}$$

This is valid for all $s \in \mathbb{R}$.

b) $G_2(s) = G(G(s))$

$$= (p\,G(s) + q)^n \quad \text{where } p = 0.6,\ q = 0.4,\ n = 2$$

$$= (0.6\,G(s) + 0.4)^2$$

$$\underline{\underline{G_2(s) = \{0.6\,(0.6s + 0.4)^2 + 0.4\}^2}}$$

c) γ is the smallest non-negative solution to $G(s) = s$.

Now $G(s) = (0.6s + 0.4)^2 = s$

$$\Rightarrow 0.36 s^2 + 0.48 s + 0.16 = s$$

$$\Rightarrow 0.36 s^2 - 0.52 s + 0.16 = 0$$

Trick: we know $s=1$ is a solution.

$$(s-1)(0.36s - 0.16) = 0$$

$$\Rightarrow s = 1 \text{ or } s = \frac{0.16}{0.36} = \frac{4}{9} = 0.44.$$

The smallest solution is $s = \frac{4}{9}$, so $\underline{\underline{\gamma = \frac{4}{9} = 0.44.}}$

3d) $\mathbb{P}(\text{extinct by generation } 4) = \mathbb{P}(Z_4 = 0)$

$$= G_4(0)$$

But $G_4(s) = G_2(G_2(s))$ (Branching process recursion)

So $G_4(0) = G_2(G_2(0))$.

From (b), $G_2(0) = \{0.6\,(0.4)^2 + 0.4\}^2$

$$= \underline{0.246}$$

So $G_4(0) = G_2(0.246)$

$$= \{0.6\,(0.6 * 0.246 + 0.4)^2 + 0.4\}^2$$

$$= \underline{0.336}$$

So $\underline{\underline{\mathbb{P}(\text{extinct by generation } 4) = G_4(0) = 0.336.}}$

3e) Each of the 10 individuals starts a new branching process, all independent; all with $\mathbb{P}(\text{extinction}) = \gamma = \frac{4}{9}$.

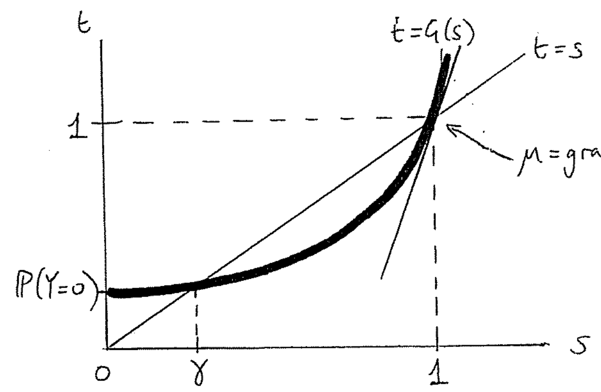
$\mathbb{P}(\text{eventual extinction}) = \mathbb{P}(\text{all 10 processes go extinct})$

$$= \gamma^{10}$$

$$= \left(\frac{4}{9}\right)^{10}$$

$$= \underline{\underline{0.00030}}.$$

4) a)



b) For $\gamma < 1$, we must have $\mu > 1$. Extinction is not certain, so on average, individuals must do more than replace themselves (i.e. $\gamma < 1 \Rightarrow \mu > 1$).

Formally, we know $\mu \leq 1 \Rightarrow \gamma = 1$

So $\gamma < 1 \Rightarrow \mu > 1$.

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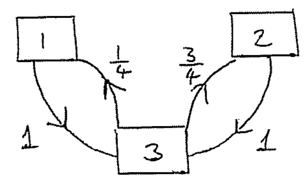
5) $$P = P^1 = \frac{1}{8}\begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 8 \\ 2 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}.$$

(putting t=1 in general solution)

a) Transition diagram:



b) $$X_1 \sim \left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\right) P = \left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\right)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$$

$$= \left(\tfrac{1}{12}\ \tfrac{1}{4}\ \tfrac{2}{3}\right)$$

So $X_1 \sim \left(\frac{1}{12}, \frac{1}{4}, \frac{2}{3}\right)^T = \left(\frac{1}{12}, \frac{3}{12}, \frac{8}{12}\right).$

5c) $\mathbb{P}(X_0 = 3,\ X_2 = 3,\ X_5 = 1)$

$= \mathbb{P}(X_0 = 3)\ \mathbb{P}(X_2 = 3 \mid X_0 = 3)\ \mathbb{P}(X_5 = 1 \mid X_2 = 3)$

$= \frac{1}{3}\ (P^2)_{33}\ (P^3)_{31}$ (*)

Putting $t=2$ in general solution for P^t:

$\Rightarrow (P^2)_{33} = \frac{1}{8}\{4 + (-1)^2 4\} = \underline{1.}$

Putting $t=3 \Rightarrow (P^3)_{31} = \frac{1}{8}\{1 + (-1)^3(-1)\} = \underline{\frac{2}{8} = \frac{1}{4}}.$

So $\mathbb{P}(X_0 = 3, X_2 = 3, X_5 = 1) = \frac{1}{3} * 1 * \frac{1}{4}$ from (*)

$= \frac{1}{12}.$

5d) From the formula for P^t, we can see that the chain does <u>not</u> converge to an equilibrium distribution as $t \to \infty$.

The dependence on t in the formula is the term $(-1)^t$, and this never vanishes as $t \to \infty$.

Thus the probability of being in a state at time t depends on whether t is even or odd, and never converges. P^{t+1} and P^t are always different as $t \to \infty$.

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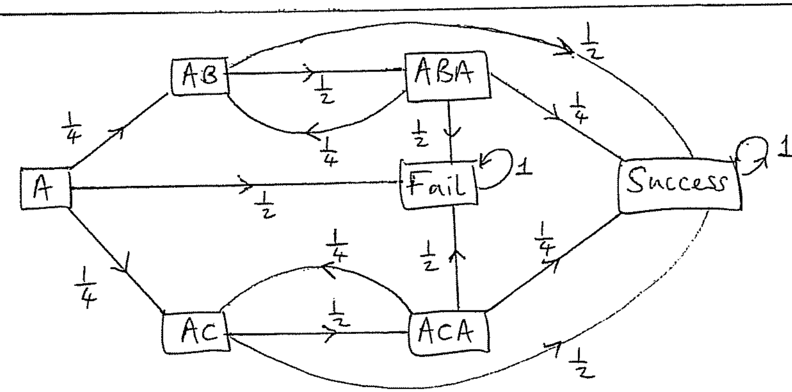
6)



6 cont) Define $p_A = \mathbb{P}(\text{success} \mid \text{start at } A)$

$p_{AB} = \mathbb{P}(\text{success} \mid \text{start at } AB)$, etc.

Then $p_A = \frac{1}{4} p_{AB} + \frac{1}{4} p_{AC} + \frac{1}{2} * 0$ (Partition Thm) $(p_{fail} = 0)$

By Symmetry, $p_{AB} = p_{AC}$.

So $p_A = \frac{1}{4} p_{AB} * 2$

$\Rightarrow p_A = \frac{1}{2} p_{AB}$ ①

Then $p_{AB} = \frac{1}{2} * 1 + \frac{1}{2} p_{ABA}$ $(p_{success} = 1)$

$\Rightarrow p_{AB} = \frac{1}{2} + \frac{1}{2} p_{ABA}$ ②

Also $p_{ABA} = \frac{1}{4} * 1 + \frac{1}{2} * 0 + \frac{1}{4} p_{AB}$ $(p_{success} = 1)$ $(p_{fail} = 0)$

$\Rightarrow p_{ABA} = \frac{1}{4} + \frac{1}{4} p_{AB}$ ③

Substitute ③ into ② for p_{ABA}:

$\Rightarrow p_{AB} = \frac{1}{2} + \frac{1}{2}\left\{\frac{1}{4} + \frac{1}{4} p_{AB}\right\}$

$\Rightarrow \frac{7}{8} p_{AB} = \frac{5}{8}$

$\Rightarrow \underline{p_{AB} = \frac{5}{7}}$.

Substitute in ① $\Rightarrow$ $\underline{\underline{p_A = \frac{1}{2} * \frac{5}{7} = \frac{5}{14}}}$ (0.357).

7a) Consider $T = \begin{cases} 1 & \text{with probability } 1/5 \\ 1 + T_1 + T_2 & \text{with prob. } \frac{4}{5}, \end{cases}$

where $T_1 \sim T$, $T_2 \sim T$, and T_1, T_2 are independent.

[T_1 = # steps to get from -1 to 0 for first time;
T_2 = # steps " " " 0 to 1 " " "].

Then $H(s) = \mathbb{E}(s^T)$

$= \frac{1}{5}\mathbb{E}(s^1) + \frac{4}{5}\mathbb{E}(s^{1+T_1+T_2})$

$= \frac{1}{5}s + \frac{4}{5}s\,\mathbb{E}(s^{T_1} s^{T_2})$

$= \frac{1}{5}s + \frac{4}{5}s\,\mathbb{E}(s^{T_1})\,\mathbb{E}(s^{T_2})$ by independence of T_1, T_2

$= \frac{1}{5}s + \frac{4}{5}s\,[H(s)]^2$ because $T_1 \sim T$ and $T_2 \sim T$ so $\mathbb{E}(s^{T_1}) = \mathbb{E}(s^{T_2}) = H(s)$.

Rearranging: $4s[H(s)]^2 - 5H(s) + s = 0$

$\Rightarrow H(s) = \dfrac{5 \pm \sqrt{25 - 4*4s*s}}{2*4s}$

$$\underline{\underline{H(s) = \frac{5 \pm \sqrt{25 - 16s^2}}{8s}}}$$

+ or − root to be determined; as stated.

b) Consider $H(0) = \mathbb{P}(T=0) = 0$, because we need at least one step to get $0 \to 1$.

Under the (+) root: $H(0) = \frac{5+5}{0}$ (undefined) $\neq 0$.
So the (+) root can not be correct.

7c) T is defective $\iff H(1) < 1$.

Now $\underline{H(1) = \dfrac{5 - \sqrt{25-16}}{8} = 0.25}$

So T $\underline{\underline{\text{is}}}$ defective (can take the value ∞).

d) $\mathbb{P}(\text{never reach state } 1) = \mathbb{P}(T = \infty)$

Now $H(s) = \sum_{t=0}^{\infty} \mathbb{P}(T=t)\, s^t$

$\Rightarrow H(1) = \sum_{t=0}^{\infty} \mathbb{P}(T=t)$ (sum of all *finite* probabilities)

So $1 = H(1) + \mathbb{P}(T=\infty)$ (sum of all probabilities of T)

$\Rightarrow \mathbb{P}(T=\infty) = 1 - H(1)$
$= 1 - 0.25$

$\underline{\underline{\mathbb{P}(T=\infty) = 0.75}}$ (prob. we never reach state 1)

e) $W = T_1 + T_2 + T_3$ where $\left.\begin{array}{l} T_1 \sim T \\ T_2 \sim T \\ T_3 \sim T \end{array}\right\}$ independent.

$$\left[\begin{array}{lll} T_1 & = \#\text{ steps } 0 \to 1 & \text{for first time} \\ T_2 & = \#\text{ steps } 1 \to 2 & \text{for first time} \\ T_3 & = \#\text{ steps } 2 \to 3 & \text{for first time} \end{array}\right]$$

So $G(s) = \mathbb{E}(s^W)$
$= \mathbb{E}(s^{T_1} s^{T_2} s^{T_3})$
$= \mathbb{E}(s^{T_1})\, \mathbb{E}(s^{T_2})\, \mathbb{E}(s^{T_3})$ because T_1, T_2, T_3 are independent

$\therefore$ $\underline{\underline{G(s) = H(s)^3}}$ because $T_1 \sim T$, $T_2 \sim T$, $T_3 \sim T$, so $\mathbb{E}(s^{T_i}) = H(s)$.

8a) i) No: T_{11} and T_{21} do <u>not</u> have the same distribution. (e.g. T_{21} can take value 1, but T_{11} can't.)

ii) Yes, T_{21} and T_{41} have the same distribution, by symmetry.

iii) Yes, T_{11} and T_{31} have the same distribution: states 1,3 both lead to states 2 and 4 with prob $\frac{1}{2}$ each, and from there it doesn't matter where we started (1 or 3) when counting # steps to get back to state 1.

8b) Consider $T_{11} = \begin{cases} 1 + T_{21} & \text{with prob. } \frac{1}{2} \\ 1 + T_{41} & \text{w.p. } \frac{1}{2} \end{cases}$

So $\mathbb{E}(s^{T_{11}}) = \frac{1}{2}\mathbb{E}(s^{1+T_{21}}) + \frac{1}{2}\mathbb{E}(s^{1+T_{41}})$
$= \frac{1}{2} s \left\{ \mathbb{E}(s^{T_{21}}) + \mathbb{E}(s^{T_{41}}) \right\}$

But $T_{21} \sim T_{41}$ by part (a)(ii),

so $\mathbb{E}(s^{T_{21}}) = \mathbb{E}(s^{T_{41}})$,

so $\mathbb{E}(s^{T_{11}}) = \frac{1}{2} s \cdot 2\,\mathbb{E}(s^{T_{21}})$

$\underline{\mathbb{E}(s^{T_{11}}) = s\,\mathbb{E}(s^{T_{21}})}$. (1)

Now consider $T_{21} = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ 1 + T_{31} & \text{w.p. } \frac{1}{2} \end{cases}$

So $\mathbb{E}(s^{T_{21}}) = \frac{1}{2}\mathbb{E}(s^1) + \frac{1}{2}\mathbb{E}(s^{1+T_{31}})$

$\underline{\mathbb{E}(s^{T_{21}}) = \frac{1}{2} s + \frac{1}{2} s\, \mathbb{E}(s^{T_{31}})}$ (2)

Here, note that $T_{31} \sim T_{11}$ from part a(iii), so $\underline{\mathbb{E}(s^{T_{31}}) = \mathbb{E}(s^{T_{11}})}$ (3)

8b cont) [Alternatively, we can derive $\mathbb{E}(s^{T_{31}}) = s\,\mathbb{E}(s^{T_{21}})$ directly, as in ①, and from there note that $T_{31} \sim T_{11}$ to complete part (a) (iii).]

Substitute ③ in ②: $\mathbb{E}(s^{T_{21}}) = \frac{1}{2}s + \frac{1}{2}s\,\mathbb{E}(s^{T_{11}})$ ④

Substitute $\mathbb{E}(s^{T_{21}})$ from ① in ④:

$$\frac{1}{s}\,\mathbb{E}(s^{T_{11}}) = \frac{1}{2}s + \frac{1}{2}s\,\mathbb{E}(s^{T_{11}})$$

$$\mathbb{E}(s^{T_{11}})\left\{\frac{2}{s} - s\right\} = s$$

$$\mathbb{E}(s^{T_{11}}) = \frac{s}{\frac{2}{s} - s}$$

$$\underline{\underline{\mathbb{E}(s^{T_{11}}) = \frac{s^2}{2 - s^2}}}$$

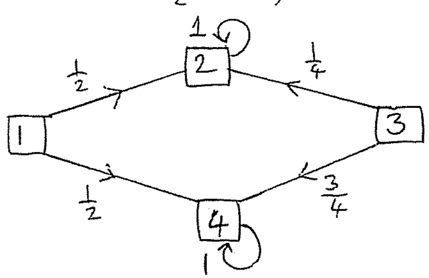
T_{11} is defective $\iff \mathbb{E}(s^{T_{11}})\big|_{s=1} < 1$.

Put $s=1$ into $\mathbb{E}(s^{T_{11}}) \Rightarrow \frac{1}{2-1} = 1$.

So T_{11} is not defective. [Also obvious from diagram.]

9a) $x = 2$ (the only state with definite hitting probability: $h_2 = 1$.)

b)



(Simplest diagram to give $\underset{\sim}{h} = (\frac{1}{2}, 1, \frac{1}{4}, 0)$.)

c) Communicating class is $\{4\}$.
Because $h_4 = 0$, state 4 can never lead to state 2.
States 1 and 3 must both lead to 2 ($h_1 > 0$ and $h_3 > 0$).

9c cont) So state 4 cannot lead to states 1 or 3 either.
So $\{4\}$ must always be a communicating class of its own.

9d) We require $m_1 = 1 + p m_2 + q m_3$

$$\Rightarrow \quad 2 = 1 + 3p + 0 * q$$

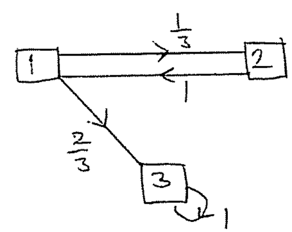
$$\Rightarrow \quad 1 = 3p \quad \text{so} \quad \underline{\underline{p = \tfrac{1}{3}}}.$$

From the diagram, $\underline{\underline{q = 1 - p = \frac{2}{3}}}$.

Similarly,
$$m_2 = 1 + r m_1 + s m_3$$
$$3 = 1 + 2r + 0$$

$\Rightarrow \underline{\underline{r = 1}}$ So also, $\underline{\underline{s = 1 - r = 0}}$.

So $\underline{\underline{p = \frac{1}{3} \quad q = \frac{2}{3} \quad r = 1 \quad s = 0}}$



10a) $Z_{n+1} = Y_1 + \dots + Y_{Z_n}$

So
$$\begin{aligned} G_{n+1}(s) &= \mathbb{E}(s^{Z_{n+1}}) \\ &= \mathbb{E}(s^{Y_1 + \dots + Y_{Z_n}}) \\ &= \mathbb{E}_{Z_n}\{\mathbb{E}(s^{Y_1 + \dots + Y_{Z_n}} \mid Z_n)\} && \text{conditional expectation} \\ &= \mathbb{E}_{Z_n}\{\mathbb{E}(s^{Y_1}) \dots \mathbb{E}(s^{Y_{Z_n}})\} && \text{because } Y_1, \dots, Y_{Z_n} \text{ are indept of each other and of } Z_n \\ &= \mathbb{E}_{Z_n}\{P(s)^{Z_n}\} \\ &= \underline{\underline{G_n(P(s)) \quad \text{as required}}}. \end{aligned}$$

10b) $G_1(s) = \mathbb{E}(s^{Z_1})$

$= \mathbb{E}(s^{Y_1+M_1})$ because $Z_1 = Y_1 + M_1$ is given

$= \mathbb{E}(s^{Y_1})\,\mathbb{E}(s^{M_1})$ by independence of Y_1, M_1

$= P(s)\,H(s)$ by definitions

$\therefore\ \underline{\underline{G_1(s) = H(s)\,P_1(s)}}$ because $P_1(s) = P(s)$.

10c) Suppose ⊛ is true for $n=k$:

$$\Rightarrow\quad G_k(s) = H(s) * H(P_1(s)) * \cdots * H(P_{k-1}(s))\,P_k(s) \qquad (**)$$

Require to prove ⊛ is true for $n=k+1$:

i.e. R.T.P. $$G_{k+1}(s) = H(s) * H(P_1(s)) * \cdots * H(P_k(s))\,P_{k+1}(s). \qquad (A)$$

$\text{LHS} = G_{k+1}(s)$

$= \mathbb{E}(s^{Z_{k+1}})$

$= \mathbb{E}(s^{Y_1+\cdots+Y_{Z_k}+M_{k+1}})$ as given for Z_{k+1}

$= \mathbb{E}(s^{M_{k+1}})\,\mathbb{E}(s^{Y_1+\cdots+Y_{Z_k}})$ by independence of M_{k+1} with $Y_1, Y_2, \ldots$ and Z_k

$= H(s)\,\mathbb{E}_{Z_k}\{\mathbb{E}(s^{Y_1+\cdots+Y_{Z_k}} \mid Z_k)\}$

$= H(s)\,G_k(P(s))$ using same arguments as in part (a)

$= H(s)\{H(P(s)) * H(P_1(P(s))) * \cdots * H(P_{k-1}(P(s)))\,P_k(P(s))\}$

substituting $P(s)$ into expression (**) for G_k

$= H(s) * H(P_1(s)) * H(P_2(s)) * \cdots * H(P_k(s))\,P_{k+1}(s)$

$= \underline{\underline{\text{RHS of (A)}}}.$

From the base case in (b), ⊛ is true when $n=1$.
From above, if ⊛ is true for $n=k$ then ⊛ is true for $n=k+1$.

So ⊛ is true for all $n = 1, 2, 3, \ldots$