STATS 325 Exam Solutions Semester 2, 2006

1a) $$\begin{aligned} LHS &= \mathbb{P}(A \cap B \mid C) \\ &= \mathbb{P}_C(A \cap B) \\ &= \mathbb{P}_C(A \mid B)\, \mathbb{P}_C(B) \\ &= \mathbb{P}(A \mid B \cap C)\, \mathbb{P}(B \mid C) \\ &= RHS \end{aligned}$$

$\Rightarrow$ TRUE

b) FALSE : We cannot add probabilities on different sample spaces meaningfully, e.g. $\mathbb{P}(A|B) + \mathbb{P}(A|C)$ does not in general make sense.

c) $$\begin{aligned} LHS &= \mathbb{P}(A|B) + \mathbb{P}(\bar{A}|B) \\ &= \mathbb{P}_B(A) + \mathbb{P}_B(\bar{A}) \\ &= 1 \\ &\neq RHS \end{aligned}$$

$\Rightarrow$ FALSE

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2) We know $(A \cap B) \subseteq A$ and $(A \cap B) \subseteq B$.
So $\mathbb{P}(A \cap B) \leqslant \mathbb{P}(A) = \frac{7}{8}$
and $\mathbb{P}(A \cap B) \leqslant \mathbb{P}(B) = \frac{1}{4}$.

Using the tighter bound, $\mathbb{P}(A \cap B) \leqslant \frac{1}{4}$.

Now consider $\mathbb{P}(A \cup B) \leqslant 1$

$$\Rightarrow \quad \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leqslant 1$$
$$\frac{7}{8} + \frac{1}{4} - \mathbb{P}(A \cap B) \leqslant 1$$
$$\Rightarrow \quad \mathbb{P}(A \cap B) \geqslant \frac{7}{8} + \frac{1}{4} - 1 = \frac{1}{8}.$$

Overall, $\frac{1}{8} \leqslant \mathbb{P}(A \cap B) \leqslant \frac{1}{4}$ as required.

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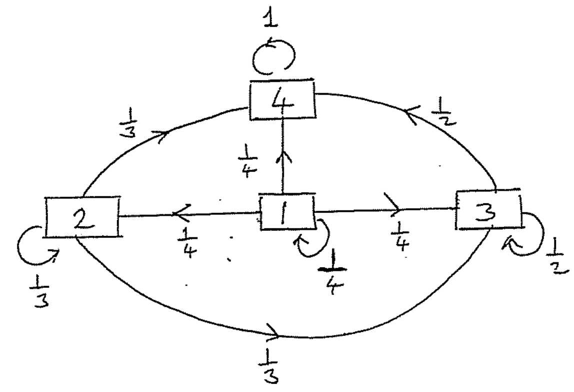
3) a)



b) Communicating classes : $\{1\}$ not closed
$\{2\}$ not closed
$\{3\}$ not closed
$\{4\}$ closed

c) Not irreducible (i.e. the chain is reducible)
Yes, all states are aperiodic because it is possible to get from each state back to itself in 1 move, so the period of each state is 1.

d) Require $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$ and $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$.

So $$(\pi_1 \ \ \pi_2 \ \ \pi_3 \ \ \pi_4) \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\pi_1 \ \ \pi_2 \ \ \pi_3 \ \ \pi_4)$$

giving $\frac{1}{4}\pi_1 = \pi_1 \Rightarrow \pi_1 = 0$

$\frac{1}{4}\pi_1 + \frac{1}{3}\pi_2 = \pi_2 \Rightarrow \frac{1}{3}\pi_2 = \pi_2$ because $\pi_1 = 0$
$\Rightarrow \pi_2 = 0$

$\frac{1}{4}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 = \pi_3 \Rightarrow \pi_3 = 0$ similarly

$$\left.\begin{aligned} \tfrac{1}{4}\pi_1 + \tfrac{1}{3}\pi_2 + \tfrac{1}{2}\pi_3 + \pi_4 &= \pi_4 \\ \text{also} \quad \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}\right\} \Rightarrow \pi_4 = 1.$$

3d cont) So the equilibrium dist^n is $\underline{\underline{\pi^T = (0, 0, 0, 1)}}$.

Yes, we can see that the chain <u>will</u> converge to this dist^n regardless of start state, because it will always hit state 4 eventually, and then get stuck there.

Note that the chain is <u>not</u> irreducible, so the general theory does not apply.

3e) By inspection, $\underline{h_{2A} = 1}$, $\underline{h_{3A} = 1}$, $\underline{h_{4A} = 0}$

Then $h_{1A} = \frac{1}{4} h_{1A} + \frac{1}{4} h_{2A} + \frac{1}{4} h_{3A} + \frac{1}{4} h_{4A}$

$\frac{3}{4} h_{1A} = \frac{1}{4} + \frac{1}{4} + 0$

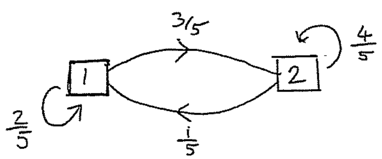
$\underline{h_{1A} = \frac{2}{3}}$

So $\underline{\underline{h_A = \left(\frac{2}{3}, 1, 1, 0\right)}}$.

4a) Using the general solution,

$$P = P^1 = \frac{1}{4}\left\{\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix}\right\} = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

Diagram :



b) $X_1 \sim \left(\frac{2}{3} \;\; \frac{1}{3}\right)\begin{pmatrix} 2/5 & 3/5 \\ 1/5 & 4/5 \end{pmatrix} = \left(\frac{1}{3} \;\; \frac{2}{3}\right)$

So $\underline{\underline{X_1 \sim \left(\frac{1}{3}, \frac{2}{3}\right)}}$.

c) $\mathbb{P}(X_0 = 2, X_2 = 2, X_5 = 1) = \mathbb{P}(X_0 = 2)\,(P^2)_{22}\,(P^3)_{21}$

$= \frac{1}{3}\,(P^2)_{22}\,(P^3)_{21}$

4c cont.) Using the general formula for P^t :

$(P^2)_{22} = \frac{1}{4}\left\{3 + \left(\frac{1}{5}\right)^2 * 1\right\} = \frac{19}{25}$

$(P^3)_{21} = \frac{1}{4}\left\{1 + \left(\frac{1}{5}\right)^3 * (-1)\right\} = \frac{31}{125}$

So $\mathbb{P}(X_0 = 2, X_2 = 2, X_5 = 1) = \frac{1}{3} * \frac{19}{25} * \frac{31}{125}$

$= \frac{589}{9375}$

$\underline{\underline{= 0.063}}$

4d) Yes, the chain <u>does</u> converge to equilibrium independent of start state as $t \to \infty$.

As $t \to \infty$, $\left(\frac{1}{5}\right)^t \to 0$ so $P^t \to \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$.

Both rows of P^t are identical as $t \to \infty$, so the equilibrium distribution is $\underline{\underline{\left(\frac{1}{4}, \frac{3}{4}\right)}}$.

5a) $\underline{\underline{p = \frac{1}{4}}}$

5b) i) T_{11} and T_{33} : yes, by symmetry.

ii) T_{12} and T_{21} : no.

iii) T_{12} and T_{32} : yes, by symmetry.

5c) $T_{31} = \begin{cases} 1 + T_{21} & \text{with probability } 3/4 \\ 1 & \text{with probability } 1/4 \end{cases}$

5c cont) So

$$\mathbb{E}(s^{T_{31}}) = \frac{3}{4}\mathbb{E}(s^{1+T_{21}}) + \frac{1}{4}\mathbb{E}(s^{1})$$

$$= \frac{3}{4}s\,\mathbb{E}(s^{T_{21}}) + \frac{1}{4}s$$

$$= \frac{3}{4}s \cdot \frac{s(4+s)}{8-3s^2} + \frac{1}{4}s \quad \text{using } \mathbb{E}(s^{T_{21}}) \text{ as given in Q}$$

$$= \frac{3s^2(4+s) + s(8-3s^2)}{4(8-3s^2)}$$

$$= \frac{12s^2 + 3s^3 + 8s - 3s^3}{4(8-3s^2)}$$

$$= \frac{12s^2 + 8s}{4(8-3s^2)}$$

$$\mathbb{E}(s^{T_{31}}) = \frac{s(3s+2)}{8-3s^2}$$

5d) $W = T_{21} + T_{11}$ by definitions, where T_{21} and T_{11} are independent.

So

$$\mathbb{E}(s^W) = \mathbb{E}(s^{T_{21}+T_{11}})$$

$$= \mathbb{E}(s^{T_{21}}\, s^{T_{11}})$$

$$= \mathbb{E}(s^{T_{21}})\,\mathbb{E}(s^{T_{11}}) \quad \text{because } T_{21} \text{ and } T_{11} \text{ are independent}$$

$$= \frac{s(4+s)}{8-3s^2} \cdot \frac{s^2(7+3s)}{2(8-3s^2)}$$

$$\mathbb{E}(s^W) = \frac{s^3(4+s)(7+3s)}{2(8-3s^2)^2}$$

6a)

$$G(s) = \mathbb{E}(s^Y) = \sum_{y=0}^{\infty} s^y\, \mathbb{P}(Y=y)$$

$$= \sum_{y=0}^{\infty} s^y\, p q^y$$

$$= p\sum_{y=0}^{\infty} (qs)^y \quad \text{(Geometric Series)}$$

$$= \frac{p}{1-qs} \quad \text{if } |qs| < 1.$$

Radius of convergence, R: we require $|qs| < 1$

$$\Rightarrow |s| < \frac{1}{q}$$

So $R = \frac{1}{q}$.

b) $\mathbb{E}(Y) = G'(1)$.

Now

$$G'(s) = \frac{d}{ds}\, p(1-qs)^{-1}$$

$$= -p(1-qs)^{-2}(-q)$$

$$= \frac{pq}{(1-qs)^2}.$$

So

$$\mathbb{E}(Y) = G'(1) = \frac{pq}{(1-q)^2}$$

$$= \frac{pq}{p^2} \quad \text{because } 1-q = p$$

$$\mathbb{E}(Y) = \frac{q}{p}.$$

c) $Y \sim \text{Geometric}(p = \frac{2}{5})$, so $q = \frac{3}{5}$.

$$G(s) = \frac{p}{1-qs} = \frac{\frac{2}{5}}{1-\frac{3}{5}s} = \frac{2}{5-3s} \quad \text{as required.}$$

6d) $$G_2(s) = G(G(s)) = \frac{2}{5-3G(s)} = \frac{2}{5-3\left(\frac{2}{5-3s}\right)} = \frac{2(5-3s)}{5(5-3s)-3*2} = \frac{10-6s}{25-15s-6}$$

$$\underline{\underline{G_2(s) = \frac{10-6s}{19-15s}}}$$

e) $\mathbb{P}(Z_1=0) = G(0) = \underline{\underline{\frac{2}{5}}}$ (0.4)

$\mathbb{P}(Z_2=0) = G_2(0) = \underline{\underline{\frac{10}{19}}}$ (0.526)

$$\mathbb{P}(Z_3=0) = G_3(0) = G(G_2(0)) = G\left(\frac{10}{19}\right) = \frac{2}{5-3*\frac{10}{19}} = \underline{\underline{\frac{38}{65}}}$$ (0.585)

$$\mathbb{P}(Z_4=0) = G_4(0) = G(G_3(0)) = G\left(\frac{38}{65}\right) = \frac{2}{5-3*\frac{38}{65}} = \underline{\frac{130}{211}}$$ (0.616)

using $G_2(G_2(0)) = G_2\left(\frac{10}{19}\right)$ is also acceptable.

6f) We require the solution to $G(s)=s$, because $\mathbb{E}(Y) = q/p = 3/2 > 1$.

$$G(s) = s \Rightarrow \frac{2}{5-3s} = s$$
$$2 = 5s - 3s^2$$
$$3s^2 - 5s + 2 = 0$$
$$(s-1)(3s-2) = 0$$
$$\Rightarrow \underline{s=1} \text{ or } \underline{s=\tfrac{2}{3}}$$

γ is the smallest solution ≥ 0, so $\underline{\underline{\gamma = \frac{2}{3}}}$.

6g) If $Y \sim \text{Poisson}\left(\frac{2}{5}\right)$, then $\mathbb{E}(Y) = \frac{2}{5} < 1$.

So $\underline{\underline{\gamma = 1}}$ because $\gamma = 1$ whenever $\mathbb{E}(Y) \leq 1$.

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7) $G_Y(s) = \mathbb{E}(s^Y)$

$= \mathbb{E}_X\{\mathbb{E}(s^Y \mid X)\}$ (Law of total expectation)

$= \mathbb{E}_X\{(ps+q)^X\}$ because $[Y \mid X] \sim \text{Bin}(X,p)$ and using Binomial PGF from the Attachment

$= G_X(ps+q)$ by definition of G_X

$= e^{\lambda(ps+q-1)}$ because $X \sim \text{Poisson}(\lambda)$ and using Poisson PGF from the Attachment

$= e^{\lambda(ps+1-p-1)}$ because $q = 1-p$

$\therefore \underline{\underline{G_Y(s) = e^{\lambda p(s-1)}}}$

This is the PGF of a Poisson(λp) distribution,

so $\underline{Y \sim \text{Poisson}(\lambda p)}$.

8) a) Wish to prove $G_n(s) = p^n s + 1 - p^n$ for $n = 1, 2, 3, \ldots$ (*)

Base case : $n=1$

$$\begin{aligned} LHS &= G_1(s) \\ &= \mathbb{E}(s^{Z_1}) \\ &= \mathbb{E}(s^Y) \quad \text{because } Z_1 \sim Y \\ &= s^0 \, \mathbb{P}(Y=0) + s^1 \, \mathbb{P}(Y=1) \\ &= 1 - p + ps \\ &= ps + 1 - p \qquad (\Delta) \\ &= \text{RHS of (*) with } n=1. \end{aligned}$$

So (*) is true for $n=1$.

General case : Suppose (*) is true for $n = 1, \ldots\ldots, x$.
So we are allowed to assume:

$$G_x(s) = p^x s + 1 - p^x \qquad (a)$$

Wish to prove (*) is true for $n = x+1$,

i.e. R.T.P. $\quad G_{x+1}(s) = p^{x+1} s + 1 - p^{x+1} \qquad (**)$

$$\begin{aligned} \text{LHS of } (**) &= G_{x+1}(s) \\ &= G_x(G(s)) \quad \text{by Branching Process recursion} \\ &= p^x G(s) + 1 - p^x \quad \text{by allowed info (a)} \\ &= p^x \{ps + 1 - p\} + 1 - p^x \quad \text{using } G(s) \text{ from } (\Delta) \\ &= p^{x+1} s + p^x - p^{x+1} + 1 - p^x \\ &= p^{x+1} s + 1 - p^{x+1} \\ &= \text{RHS of } (**). \end{aligned}$$

So we have proved (*) true for $n=1$,
and if true for $n=x$ then true for $n=x+1$,
thus (*) is true for all $n = 1, 2, 3, \ldots$ as required.

8b) $$\begin{aligned} \mathbb{P}(\text{extinct } \underline{\text{by}} \text{ generation } n) &= \mathbb{P}(Z_n = 0) \\ &= G_n(0) \\ &= p^n * 0 + 1 - p^n \\ &= 1 - p^n \quad \text{as stated.} \end{aligned}$$

Explanation :
At each generation, there is at most one individual :
- it survives to next generation with probability p,
- it dies with prob. $1-p$.

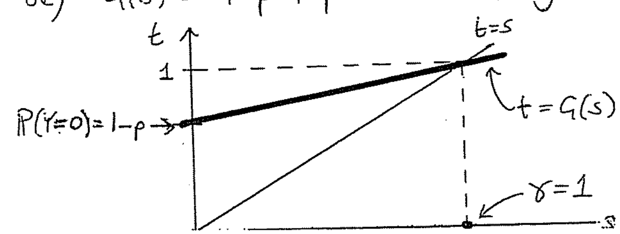
This is because $\quad Y = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{with prob } 1-p. \end{cases}$

So $$\begin{aligned} \mathbb{P}(\text{extinct by gen}^n\ n) &= 1 - \mathbb{P}(\text{still alive by gen}^n\ n) \\ &= 1 - \mathbb{P}(\text{survived } n \text{ times}) \\ &= 1 - p^n. \end{aligned}$$

8c) $$\begin{aligned} \mathbb{P}(\text{extinct } \underline{\text{at}} \text{ generation } n) &= \mathbb{P}(Z_n = 0) - \mathbb{P}(Z_{n-1} = 0) \\ &= (1 - p^n) - (1 - p^{n-1}) \\ &= 1 - p^n - 1 + p^{n-1} \\ &= p^{n-1}(1-p). \end{aligned}$$ [survives $n-1$ times, then dies]

8d) $\mathbb{E}(Y) = 1 * p + 0 * (1-p) = p$ and we are told $p < 1$,
So
$\mathbb{E}(Y) < 1 \quad \therefore \quad \gamma = 1.$

8e) $G(s) = 1 - p + ps$ is a straight line with slope $p < 1$.



9) a) $k=0$:

$$\begin{aligned} \text{LHS of } (*) &= h_0 = 1 \quad \text{(by inspection)} \\ \text{RHS of } (*) &= (2^0-1)h_1 - (2^0-2) \\ &= 0*h_1 - 1 + 2 \\ &= 1 \\ &= \text{LHS of } (*) \end{aligned}$$

So $(*)$ is true for $k=0$.

$k=1$:

$$\begin{aligned} \text{LHS of } (*) &= h_1 \\ \text{RHS of } (*) &= (2-1)h_1 - (2-2) \\ &= h_1 \\ &= \text{LHS of } (*) \end{aligned}$$

So $(*)$ is true for $k=1$.

b) Hitting probabilities: $h_r = \frac{2}{3}h_{r-1} + \frac{1}{3}h_{r+1}$ $(r=1,2,3,\ldots)$

$$\Rightarrow \quad 3h_r = 2h_{r-1} + h_{r+1}$$

$$\Rightarrow \quad h_{r+1} = 3h_r - 2h_{r-1} \quad \text{as stated.}$$

Valid for $r=1,2,3,\ldots$

c) We may assume: (i) $h_k = (2^k-1)h_1 - (2^k-2)$ for $k=0,1,\ldots,r$

(ii) $h_{r+1} = 3h_r - 2h_{r-1}$

Wish to prove $(*)$ for $k=r+1$, i.e.

R.T.P. $\quad h_{r+1} = (2^{r+1}-1)h_1 - (2^{r+1}-2) \quad (**)$

$$\begin{aligned} \text{LHS of } (**) &= h_{r+1} \\ &= 3h_r - 2h_{r-1} \quad \text{by allowed info (ii)} \\ &= 3\{(2^r-1)h_1 - (2^r-2)\} - 2\{(2^{r-1}-1)h_1 - (2^{r-1}-2)\} \end{aligned}$$

by allowed info (i) for $k=r$ and $k=r-1$

9c cont) So

$$\begin{aligned} \text{LHS of } (**) &= h_1\{3*2^r - 3 - 2*2^{r-1} + 2\} \\ &\quad - 3*2^r + 3*2 + 2*2^{r-1} - 2*2 \\ &= h_1\{3*2^r - 2^r - 1\} - 3*2^r + 2^r + 2 \\ &= h_1\{2^r(3-1) - 1\} + 2^r(-3+1) + 2 \\ &= h_1\{2^r*2 - 1\} + 2^r(-2) + 2 \\ &= h_1\{2^{r+1} - 1\} - 2^{r+1} + 2 \\ &= (2^{r+1}-1)h_1 - (2^{r+1}-2) \\ &= \text{RHS of } (**) \end{aligned}$$

So if $(*)$ is true for $k=0,\ldots,r$, then $(*)$ is true for $k=r+1$.

We proved $(*)$ true for $k=0$ and $k=1$ in (a).

So $(*)$ is true for all $k=0,1,2,3,\ldots$

9d) We have $\quad 0 \le h_k \le 1 \quad$ for all $k=0,1,2,\ldots$

$$\Rightarrow \quad 0 \le (2^k-1)h_1 - (2^k-2) \le 1 \quad \text{using } h_k \text{ from } (*)$$

Use lower limit: $\quad 0 \le (2^k-1)h_1 - (2^k-2)$

$$\Rightarrow \quad h_1 \ge \frac{2^k-2}{2^k-1} \quad \text{for all } k=0,1,2,\ldots$$

As $k \to \infty$, we get $h_1 \ge \lim_{k\to\infty}\left(\frac{2^k-2}{2^k-1}\right) = 1$.

Because h_1 is a probability, we also need $h_1 \le 1$.
So the only possible value is $h_1 = 1$.

9d cont) Then $h_k = (2^k - 1)h_1 - (2^k - 2)$ for $k = 0, 1, 2, \dots$

$$= (2^k - 1) * 1 - 2^k + 2$$

$$= 2^k - 2^k - 1 + 2$$

$$\therefore \underline{\underline{h_k = 1 \quad \text{for all } k = 0, 1, 2, \dots}}$$

9e) $h_{10} = \mathbb{P}(\text{hit state } 0, \text{ starting from state } 10) = 1$ by above.

So we definitely hit 0 starting from 10, in other words we will always get there eventually.

Now $T = \infty \iff$ we never reach 0 starting from 10

So $\mathbb{P}(T = \infty) = 0$
Therefore, $\underline{\underline{T \text{ is not defective.}}}$

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10) $$\mathbb{P}(W < T) = \mathbb{E}_T\{\mathbb{P}(W < T \mid T)\}$$

$$= \int_{t=0}^{\infty} \mathbb{P}(W < T \mid T = t)\, f_T(t)\, dt$$

$$= \int_{t=0}^{\infty} \mathbb{P}(W < t)\, f_T(t)\, dt$$

[because W, T are indept, so $\mathbb{P}(W < T \mid T = t) = \mathbb{P}(W < t)$]

$$= \int_{t=0}^{\infty} F_W(t)\, f_T(t)\, dt$$

$$= \int_{t=0}^{\infty} (1 - e^{-8t}) . 6e^{-6t}\, dt$$

because $W \sim Exp(8)$, $T \sim Exp(6)$

$$= \int_{t=0}^{\infty} (6e^{-6t} - 6e^{-6t}e^{-8t})\, dt$$

10 cont) So $$\mathbb{P}(W < T) = \int_{t=0}^{\infty} (6e^{-6t} - 6e^{-14t})\, dt$$

$$= \left[-e^{-6t} + \frac{6}{14}e^{-14t}\right]_0^{\infty}$$

$$= -0 + 0 - \left\{-e^0 + \frac{6}{14}e^0\right\}$$

$$= -\left\{-1 + \frac{6}{14}\right\}$$

$$\therefore \underline{\underline{\mathbb{P}(W < T) = \frac{4}{7}}}$$
