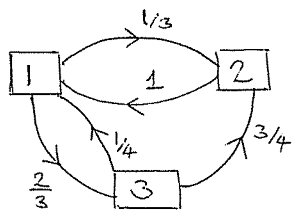
STATS 325 Exam Solutions Semester 2, 2008

1a)



b) Single communicating class : $\{1, 2, 3\}$.
The class is closed.

c) Require $\underset{\sim}{\pi}$ such that $\underset{\sim}{\pi}^T P = \pi^T$ and $\pi_1 + \pi_2 + \pi_3 = 1$.

Consider $\underset{\sim}{\pi}^T P = (\pi_1 \ \pi_2 \ \pi_3) \begin{pmatrix} 0 & 1/3 & 2/3 \\ 1 & 0 & 0 \\ 1/4 & 3/4 & 0 \end{pmatrix} = (\pi_1 \ \pi_2 \ \pi_3)$.

Equations: $\pi_2 + \frac{1}{4}\pi_3 = \pi_1$ (1)

[skip #2]

$\frac{2}{3}\pi_1 = \pi_3$ (3)

$\pi_1 + \pi_2 + \pi_3 = 1$ (4)

(3) in (1) $\Rightarrow$ $\pi_2 = \pi_1 - \frac{1}{4}\cdot\frac{2}{3}\pi_1 = \frac{5}{6}\pi_1$

In (4) $\Rightarrow$ $\pi_1 \left\{ 1 + \frac{5}{6} + \frac{2}{3} \right\} = 1$

$\underline{\pi_1 = \frac{2}{5}}$

$\therefore$ $\underline{\pi_3 = \frac{2}{3}\cdot\frac{2}{5} = \frac{4}{15}}$, $\underline{\pi_2 = \frac{5}{6}\cdot\frac{2}{5} = \frac{5}{15}}$

So $\underline{\underline{\underset{\sim}{\pi} = \left(\frac{6}{15}, \frac{5}{15}, \frac{4}{15}\right) = (0.4, 0.33, 0.27)}}$

1d) The chain $\{X_t\}$ is irreducible and aperiodic, and an equilibrium distribution exists.
So the chain X_t <u>does</u> converge to equilibrium $\underset{\sim}{\pi}$ as $t \to \infty$.

e) $\underset{\sim}{m}$ satisfies : $m_3 = 0$

$m_1 = 1 + \frac{2}{3} * 0 + \frac{1}{3} m_2$ (a)

$m_2 = 1 + m_1$ (b)

(b) in (a) : $m_1 = 1 + \frac{1}{3}(1 + m_1)$

$\frac{2}{3} m_1 = \frac{4}{3}$

$\underline{m_1 = 2}$

So $\underline{\underline{\underset{\sim}{m} = (m_1, m_2, m_3) = (2, 3, 0)}}$.

2a) $G_T(s) = \mathbb{E}(s^T)$

$= \mathbb{E}(s^{X_1 + \dots + X_N})$

$= \mathbb{E}_N \left\{ \mathbb{E}(s^{X_1 + \dots + X_N} \mid N) \right\}$

$= \mathbb{E}_N \left\{ \mathbb{E}(s^{X_1 + \dots + X_N}) \right\}$ where N is treated as constant within the inner term: by indep of $X_1, X_2, \dots$ and N

$= \mathbb{E}_N \left\{ \mathbb{E}(s^{X_1}) \dots \mathbb{E}(s^{X_N}) \right\}$ by indep of $X_1, X_2, \dots$

$= \mathbb{E}_N \left\{ G_X(s)^N \right\}$ by definition of $G_X(s)$

$= G_N(G_X(s))$ by definition of $G_N(s)$.

So $\underline{\underline{G_T(s) = G_N(G_X(s))}}$ as stated.

2b) $G_X(s) = \mathbb{E}(s^X)$

$= s^0\,\mathbb{P}(X=0) + s^1\,\mathbb{P}(X=1)$

$\therefore\ \underline{\underline{G_X(s) = 1-p+ps}}$

c) $N \sim \text{Poisson}(\lambda)$

so $G_N(r) = e^{\lambda(r-1)}$ (*)

Thus $G_T(s) = G_N(G_X(s))$

$= G_N(1-p+ps)$ from (b)

$= e^{\lambda(1-p+ps-1)}$ setting $r = 1-p+ps$ in (*)

$= e^{\lambda(ps-p)}$

$\underline{\underline{G_T(s) = e^{\lambda p(s-1)}}}$ PGF of T.

This is the PGF of the Poisson(λp) distribution.

So $\underline{\underline{T \sim \text{Poisson}(\lambda p)}}$.

3a) i)

$$P = \begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & \frac14 & \frac14 & \frac14 & \frac14 \\ 0 & \frac14 & \frac14 & \frac14 & \frac14 \\ 0 & 0 & \frac12 & \frac14 & \frac14 \\ 0 & 0 & 0 & \frac34 & \frac14 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\end{array}$$

ii)

$$P = \begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \frac14 & \frac34 & 0 & 0 \\ 0 & 0 & \frac12 & \frac12 & 0 \\ 0 & 0 & 0 & \frac34 & \frac14 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\end{array}$$

iii)

$$P = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\ \vdots \end{array}\begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots \end{array} \\ \begin{pmatrix} \frac34 & \frac14 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac34 & \frac14 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac34 & \frac14 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac34 & \frac14 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac34 & \frac14 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \frac34 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\end{array}$$

3b) $\mathbb{E}T$ is the expected reaching time for state 4 in Chain (ii), starting from state 0.

3b cont) Let $\underset{\sim}{m} = (m_0, m_1, \ldots, m_4)$ be the vector of expected reaching times for state 4 in chain (ii).

We require $\mathbb{E}(T) = m_0$.

Now $m_4 = 0$

$m_3 = 1 + \frac{3}{4}m_3 + \frac{1}{4} * 0 \Rightarrow \frac{1}{4}m_3 = 1$

$\Rightarrow \underline{m_3 = 4}$

$m_2 = 1 + \frac{1}{2}m_2 + \frac{1}{2}m_3 \Rightarrow \frac{1}{2}m_2 = 1 + \frac{4}{2}$

$\Rightarrow \underline{m_2 = 6}$

$m_1 = 1 + \frac{1}{4}m_1 + \frac{3}{4}m_2 \Rightarrow \frac{3}{4}m_1 = 1 + \frac{3}{4} * 6$

$\Rightarrow \underline{m_1 = \frac{22}{3}}$

$m_0 = 1 + m_1 \Rightarrow \underline{m_0 = \frac{25}{3} = 8.33}$

Thus $\underline{\underline{\mathbb{E}(T) = m_0 = 8.33 \text{ packets}}}$

4a) $G(s) = \mathbb{E}(s^Y) = \sum_{y=0}^{2} s^y \, \mathbb{P}(Y=y)$

$= \sum_{y=0}^{2} \binom{2}{y} \left(\frac{4s}{5}\right)^y \left(\frac{1}{5}\right)^{2-y}$

$= \left(\frac{4s}{5} + \frac{1}{5}\right)^2$ (Binomial Theorem)

$\underline{\underline{G(s) = \frac{1}{25}(4s+1)^2}}$ $\underline{\underline{\text{Valid for } s \in \mathbb{R}}}$

as stated.

4b) $G_2(s) = G(G(s))$

$= \frac{1}{25}(4G(s) + 1)^2$

$\underline{\underline{G_2(s) = \frac{1}{25}\left\{\frac{4}{25}(4s+1)^2 + 1\right\}^2}}$

c) $\mathbb{P}(Z_4 = 0) = G_4(0)$

$= G_2(G_2(0))$

Now $G_2(0) = \frac{1}{25}\left\{\frac{4}{25} + 1\right\}^2 = \frac{841}{15625}$

So $G_2(G_2(0)) = \frac{1}{25}\left\{\frac{4}{25}\left(\frac{4 * 841}{15625} + 1\right)^2 + 1\right\}^2$

$\underline{= 0.061}$

So $\underline{\underline{\mathbb{P}(Z_4 = 0) = 0.061}}$ as stated.

d) γ is the smallest non-negative solution to the equation $G(s) = s$.

Consider $G(s) = s$

$\frac{1}{25}(4s+1)^2 = s$

$\Rightarrow 16s^2 + 8s + 1 = 25s$

$16s^2 - 17s + 1 = 0$

$(s-1)(16s-1) = 0$

$\Rightarrow s = 1, \; s = \frac{1}{16}$.

So γ is the smaller solution $\Rightarrow \underline{\underline{\gamma = \frac{1}{16}}}$.

4e) If $Z_2 = 5$, we have 5 independent branching processes which must all go extinct.

$$\mathbb{P}(\text{extinction}) = \gamma^5 = \left(\frac{1}{16}\right)^5 = 9.5 \times 10^{-7}.$$

f) From (c), $\mathbb{P}$(one indiv has living descendents 4 generations later)

$$= 1 - \mathbb{P}(Z_4 = 0)$$
$$= 1 - 0.061$$
$$= 0.939.$$

So for 5 independent individuals in generation 2,

$\mathbb{P}$(all five have living descendents in generation 6)

$$= (0.939)^5$$
$$= 0.730.$$

5a) Consider $T = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ 1 + T' + T'' & \text{w.p. } \frac{1}{2} \end{cases}$

where $T' \sim T'' \sim T$
and T', T'' are independent.

Thus $H(s) = \mathbb{E}(s^T) = \frac{1}{2}s + \frac{1}{2}s\,\mathbb{E}(s^{T'+T''})$

$= \frac{1}{2}s + \frac{1}{2}s\,\mathbb{E}(s^{T'})\,\mathbb{E}(s^{T''})$ (indep of T', T'')

$\therefore H(s) = \frac{1}{2}s + \frac{1}{2}s\left[H(s)\right]^2$ because $T' \sim T'' \sim T$

$$\Rightarrow s\left[H(s)\right]^2 - 2H(s) + s = 0$$

5 a cont) Solving:

$$H(s) = \frac{2 \pm \sqrt{4 - 4s^2}}{2s}$$

$$\therefore H(s) = \frac{1 \pm \sqrt{1 - s^2}}{s} \quad \text{as stated.}$$

5b) $H(0) = \mathbb{P}(T = 0) = 0$ because we require ≥ 1 step to go from state 0 to state 1.

For the (+) root: $H(0) = \frac{1+1}{0}$ (undefined) $\neq 0$.

Thus the (+) root can not be correct.

c) T is defective $\iff H(1) < 1$.

Now $H(1) = \frac{1 - \sqrt{1-1}}{1} = 1$.

So T is not defective.

d) $\mathbb{E}(T) = H'(1)$

Now $H(s) = s^{-1} - \sqrt{\frac{1}{s^2} - 1}$

$= s^{-1} - \left(s^{-2} - 1\right)^{1/2}$

So $H'(s) = -s^{-2} - \frac{1}{2}\left(s^{-2} - 1\right)^{-1/2}\left(-2s^{-3}\right)$

$H'(s) = -\frac{1}{s^2} + \frac{1}{s^3\sqrt{\frac{1}{s^2} - 1}}$ ← tends to 0 as $s \to 1$

So $H'(1) = \infty$

$\therefore \mathbb{E}(T) = \infty.$

5e) $\mathbb{P}(\text{never reach } 1 \text{ from } 0) = \mathbb{P}(T=\infty)$

$= 0$ because T is not defective.

f) By inspection:

$p_0 = 0$ (Can't go $0 \to 1$ in 0 steps)

$p_1 = \frac{1}{2}$ ($0 \to 1$ in 1 step)

$p_2 = 0$ (Can't go $0 \to 1$ in 2 steps)

Now consider
$$T \sim \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 1 + T' + T'' & \text{w.p. } \frac{1}{2} \end{cases}$$
where $T' \sim T'' \sim T$
T', T'' indept.

Thus for $t \geq 3$ we have:

$$\mathbb{P}(T=t) = \tfrac{1}{2}\mathbb{P}(1+T'+T''=t)$$
$$= \tfrac{1}{2}\mathbb{P}(T'+T''=t-1) \quad \text{where } T' \sim T'' \sim T \text{, } T',T'' \text{ indept}$$
$$= \tfrac{1}{2}\sum_{x=0}^{t-1} \mathbb{P}(T'+T''=t-1 \mid T'=x)\,\mathbb{P}(T'=x) \quad \text{Partition Thm.}$$
$$= \tfrac{1}{2}\sum_{x=0}^{t-1} \mathbb{P}(T''=t-1-x)\,\mathbb{P}(T'=x) \quad \text{indep of } T',$$

$$\therefore\ \mathbb{P}(T=t) = \tfrac{1}{2}\sum_{x=0}^{t-1} p_{t-1-x}\, p_x$$
by definition of p_{t-1-x} and p_x, because $T' \sim T'' \sim T$.

Valid as stated for $t = 3, 4, 5, \ldots$

6) a) From the diagram:

$$h_x = \tfrac{3}{5}h_{x+1} + \tfrac{2}{5}h_{x-1} \quad \text{for } x = 1,2,3,\ldots$$

So $3h_{x+1} - 5h_x + 2h_{x-1} = 0$ for $x = 1,2,3,\ldots$

as stated.

Boundary: $h_0 = 1$.

b) Consider $3t^2 - 5t + 2 = 0$

$(3t-2)(t-1) = 0$

$\therefore\ t_1 = \frac{2}{3} \qquad t_2 = 1$

$\therefore$ General solution of (*) is

$$h_x = A\left(\tfrac{2}{3}\right)^x + B \qquad x = 0,1,2,\ldots \quad (1)$$

Boundary: $h_0 = A + B = 1$

$\Rightarrow B = 1 - A$

In (1) $\Rightarrow h_x = A\left(\frac{2}{3}\right)^x + 1 - A$

$$h_x = 1 - A\left(1 - \left(\tfrac{2}{3}\right)^x\right) \qquad x = 0,1,2,\ldots$$

as stated.

c) We need the minimal h_x:

$$h_x = 1 - A\underbrace{\left(1 - \left(\tfrac{2}{3}\right)^x\right)}_{\text{always} \geq 0}$$

6c cont) So $h_x = 1 - A * (\text{something positive})$

so the minimum h_x occurs with the maximum A.

Now consider h_x at the extremes of x:

$x = 0$: $0 \leq h_0 \leq 1$ because h_0 is a probability

(no help because formula gives $h_0 = 1$ for all A)

$x = \infty$: $0 \leq h_\infty \leq 1$

$\Rightarrow \quad 0 \leq 1 - A(1-0) \leq 1$

$0 \leq 1 - A \leq 1$

$\Rightarrow \quad \underline{A \leq 1}$ and $\underline{A \geq 0}$

So for the maximum A and because $A \leq 1$, we must have $\underline{\underline{A = 1}}$.

So
$$h_x = 1 - A\left(1 - \left(\tfrac{2}{3}\right)^x\right)$$
$$= 1 - 1\left(1 - \left(\tfrac{2}{3}\right)^x\right) \quad \text{using } A = 1$$
$$\underline{\underline{h_x = \left(\tfrac{2}{3}\right)^x \quad \text{for all } x = 0, 1, 2, \ldots}}$$

6d) It would not be possible to prove $h_x = \left(\frac{2}{3}\right)^x$ by mathematical induction, because we would need two base cases ($x=0$ and $x=1$) to use the 2nd-order allowed info ⊛ : $h_{x+1} = \frac{1}{3}\{5h_x - 2h_{x-1}\}$. As we only have one base case ($x=0$: $h_0 = 1$) we cannot establish the inductive base.

6d cont) Note (non-examinable): it would be possible to establish an induction for an expression relating h_x to h_1. Then the base cases $h_0 = 1$ and $h_1 = h_1$ suffice. However, this expression is complicated to obtain and manipulate. It is:
$$h_x = 3h_1\left(1 - \left(\tfrac{2}{3}\right)^x\right) + 3\left(\tfrac{2}{3}\right)^x - 2.$$
Then we still need the minimality of h_x to obtain h_1.

6e) Yes, T is defective, because
$$h_1 = \mathbb{P}(\text{eventually reach } 0 \text{ from } 1) = \tfrac{2}{3} < 1.$$
So we might never reach state 0 starting from state 1, so $T = \#\text{steps needed}$ can take the value $T = \infty$.

7a) $a = G(0) = G_1(0) = \gamma_1$ by hint.

$b = a = \gamma_1$

$c = G(b) = G(G(0)) = G_2(0) = \gamma_2$ by hint.

$d = c = \gamma_2$

$e = \gamma$ (smallest non-negative solution of $G(s) = s$).

So $\underline{\underline{a = \gamma_1 \quad b = \gamma_1 \quad c = \gamma_2 \quad d = \gamma_2 \quad e = \gamma}}$.

b) μ is the gradient of $t = G(s)$ at the point $s = 1$.
The curve $t = G(s)$ is steeper than the line $t = s$ (gradient $= 1$)

So $\underline{\underline{\mu > 1}}$.

7c) Require to prove $\gamma_n = \dfrac{\mu^n - 1}{\mu^{n+1} - 1}$ $\quad n = 1, 2, 3, \dots$ $\circledast$.

<u>Base case :</u> $n = 1$

$$\text{LHS of } \circledast = \gamma_1 = G_1(0) = \frac{1}{\mu + 1} \quad \text{by info in question}$$

$$\text{RHS of } \circledast = \frac{\mu - 1}{\mu^2 - 1} = \frac{\mu - 1}{(\mu+1)(\mu-1)} = \frac{1}{\mu+1} = \underline{\text{LHS of } \circledast} .$$

So <u>base case $n=1$ is proved.</u>

<u>General case:</u> suppose $\circledast$ is true for $n = x$ for some $x \geq 1$, $x \in \mathbb{Z}$

So we may assume that

$$\gamma_x = \frac{\mu^x - 1}{\mu^{x+1} - 1} \qquad (a)$$

R.T.P. $\circledast$ is true for $n = x+1$,

i.e. R.T.P. $\gamma_{x+1} = \dfrac{\mu^{x+1} - 1}{\mu^{x+2} - 1}$ $\quad (**)$.

7c cont) LHS of $(**)$

$$\begin{aligned}
&= \gamma_{x+1} \\
&= G_{x+1}(0) && \text{(info in question)} \\
&= G(G_x(0)) && \text{(Branching process recursion)} \\
&= G(\gamma_x) && \text{(info in Q : } \gamma_x = G_x(0)) \\
&= \frac{1}{\mu + 1 - \mu\gamma_x} && \text{(form for } G(s) \text{ given in Q)} \\
&= \frac{1}{\mu + 1 - \mu\left(\frac{\mu^x - 1}{\mu^{x+1} - 1}\right)} && \text{using allowed info (a)} \\
&= \frac{\mu^{x+1} - 1}{(\mu^{x+1} - 1)(\mu + 1) - \mu(\mu^x - 1)} && \left(\text{mult. by } \mu^{x+1} - 1 \text{ everywhere}\right) \\
&= \frac{\mu^{x+1} - 1}{\mu^{x+2} + \cancel{\mu^{x+1}} - \cancel{\mu} - 1 - \cancel{\mu^{x+1}} + \cancel{\mu}} && \text{expanding} \\
&= \frac{\mu^{x+1} - 1}{\mu^{x+2} - 1} \\
&= \underline{\text{RHS of } (**)}
\end{aligned}$$

So if $\circledast$ is true for $n = x$, it is proved true for $n = x+1$.

We proved $\circledast$ true for base case $n = 1$.

Thus $\circledast$ is proved true for all $n = 1, 2, 3, \dots$.