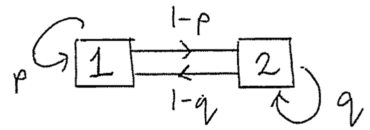
STATS 325 — Exam Solutions — Semester 2, 2009

1a) 

b) Single communicating class is $\{1, 2\}$: closed.

c) Let $\underset{\sim}{\pi}^T = (\pi_1, \pi_2)$. We require $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$ and $\pi_1 + \pi_2 = 1$

Now $\underset{\sim}{\pi}^T P = (\pi_1 \ \ \pi_2)\begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix} = (\pi_1 \ \ \pi_2)$

$$\Rightarrow \quad p\pi_1 + (1-q)\pi_2 = \pi_1 \quad \text{(Gain 1 eqn only)} \quad (a)$$

Also $$\pi_1 + \pi_2 = 1 \quad \Rightarrow \quad \pi_2 = 1 - \pi_1 \quad (b)$$

Subst (b) in (a) $$\Rightarrow \quad p\pi_1 + (1-q)(1-\pi_1) = \pi_1$$

$$1-q = \pi_1\{1 + 1 - q - p\}$$

$$\therefore \quad \pi_1 = \frac{1-q}{2-p-q}$$

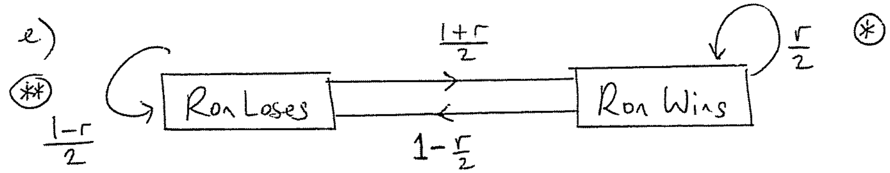
$$\therefore \quad \pi_2 = 1 - \pi_1 = \frac{2-p-q-1+q}{2-p-q} = \frac{1-p}{2-p-q}$$

So the equilibrium distribution is

$$\underset{\sim}{\pi}^T = \left(\frac{1-q}{2-p-q}, \ \frac{1-p}{2-p-q}\right) \quad \text{as stated.}$$

(Note $p<1$, $q<1$ given in Q!)

1d) The chain is irreducible and aperiodic, and an equilibrium distribution $\underset{\sim}{\pi}$ exists, so the chain does converge to $\underset{\sim}{\pi}$ as $t \to \infty$.

e) 

(*) From Wins: only way Ron can win again is if he has red & computer has blue: probability $r * \frac{1}{2} = \frac{r}{2}$.

(**) From Loses: only way Ron can lose again is if he has blue & computer has red: probability $(1-r) * \frac{1}{2} = \frac{1-r}{2}$.

The other two probabilities are found by subtracting from 1.

f) We have $p = \frac{1-r}{2}$, $q = \frac{r}{2}$, in notation of part (c).

So $$\underset{\sim}{\pi}^T = \left(\frac{1 - r/2}{2 - \left(\frac{1-r}{2}\right) - \frac{r}{2}}, \ \frac{1 - \frac{1-r}{2}}{2 - \left(\frac{1-r}{2}\right) - \frac{r}{2}}\right)$$

$$= \left(\frac{2-r}{4-1+r-r}, \ \frac{2-1+r}{4-1+r-r}\right)$$

$$\underset{\sim}{\pi}^T = \left(\frac{2-r}{3}, \ \frac{1+r}{3}\right)$$

g) From (d), the chain converges to $\underset{\sim}{\pi}$. So long-run proportion of time in state 2 (Ron Wins) is $\pi_2 = \frac{1+r}{3}$ as stated.

1h) Wish to maximise $\frac{1+r}{3}$ with respect to r, where $0<r<1$ (a probability).

This is a monotone increasing function of r, so we need $\underline{\underline{r=1}}$

i.e. Ron should always choose Red.

2a) $Y \sim \text{Binomial}(3, \frac{1}{2})$:

y	0	1	2	3
$P(Y=y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(using Formula Sheet or general knowledg

So $G(s) = \mathbb{E}(s^Y)$

$$= \frac{1}{8}s^0 + \frac{3}{8}s^1 + \frac{3}{8}s^2 + \frac{1}{8}s^3$$

$\therefore G(s) = \frac{1}{8}\{s^3 + 3s^2 + 3s + 1\}$ as stated.

b) $\mathbb{P}(Z_2 = 0) = G(G(0))$

$$= G(\tfrac{1}{8})$$

$= \underline{\underline{0.178}}$ using $G(s)$ in (a) with $s = \frac{1}{8}$.

c) $\mathbb{P}(\text{not extinct by generation } 2) = 1 - \mathbb{P}(Z_2 = 0)$

$$= 1 - 0.178$$

$$= \underline{\underline{0.822}}$$

d) γ is smallest solution ≥ 0 to $G(s) = s$.

2d cont.) Require $s = G(s)$

$$\Rightarrow \quad 8s = s^3 + 3s^2 + 3s + 1$$

$$\Rightarrow \quad s^3 + 3s^2 - 5s + 1 = 0$$

Know $(s-1)$ is a factor:

$(s-1)(s^2 + 4s - 1) = 0$ factorizing

$s = 1$ or $s = \frac{-4 \pm \sqrt{16+4}}{2} = 0.236$, 2nd root something negative

So required solution is $\underline{\underline{\gamma = 0.236}}$.

2e) $\mathbb{P}(\text{eventual extinction} \mid Z_2 = 3) = \gamma^3 = (0.236)^3 = \underline{\underline{0.013}}$.

3a) $\mu < 1$: $\underline{\underline{\text{C only}}}$

Require $t = G(s)$ to be above $t = s$ at $s = 1$ so that gradient is $\mu < 1$.

b) $\mu > 1$: $\underline{\underline{\text{A and B}}}$

Require $t = G(s)$ forced below $t = s$ at $s = 1$ due to gradient μ steeper than 1.

c) $\gamma = 1$: $\underline{\underline{\text{C only}}}$: require 1st intersection between $t = s$ and $t = G(s)$ to be at $s = 1$, for $s \geq 0$

d) $\mathbb{P}(Y > 0) = 1$: $\underline{\underline{\text{B only}}}$: require $t = G(s)$ to intersect t-axis at $G(0) = \mathbb{P}(Y = 0) = 0$.

e) $G(1) = 1$: $\underline{\underline{\text{A, B, C}}}$. True for all non-defective r.v.'s Y.

(5)

4a) First-step analysis

$$\Rightarrow h_x = 0.48\, h_{x+1} + 0.48\, h_{x-1} + 0.04 * 0$$

$$\Rightarrow \underline{\underline{0.48 h_{x+1} - h_x + 0.48\, h_{x-1} = 0}} \quad \text{as stated;}$$

for $x = 2, 3, \ldots, 19$.

b) $\underline{\underline{h_{20} = 1}}$ (1)

$h_1 = 0.48\, h_1 + 0.48 h_2$ by first-step analysis

$\Rightarrow \underline{\underline{0.52\, h_1 = 0.48\, h_2}}$ (2) (best we can do).

c) $0.48\, t^2 - t + 0.48 = 0$

$$\Rightarrow t = \frac{1 \pm \sqrt{1 - 4*(0.48)^2}}{2*0.48} = \frac{1 \pm 0.28}{0.96}$$

$$\therefore \underline{\underline{t_1 = \frac{3}{4}}} \quad \text{and} \quad \underline{\underline{t_2 = \frac{4}{3}}}$$

Thus general solution to (*) is:

$$\underline{\underline{h_x = A\left(\frac{3}{4}\right)^x + B\left(\frac{4}{3}\right)^x, \quad x = 1, \ldots, 20.}}$$

d) Using (2) from (b):

$$0.52\, h_1 = 0.48\, h_2$$

$$\therefore A\left(\frac{3}{4}\right) + B\left(\frac{4}{3}\right) = \frac{12}{13}\left\{A\left(\frac{3}{4}\right)^2 + B\left(\frac{4}{3}\right)^2\right\}$$

$$A\left(\frac{3}{4} - \frac{27}{52}\right) = B\left(\frac{4}{13}\right)$$

$$\Rightarrow \underline{\underline{B = \frac{3}{4}A}} \quad \text{as stated.}$$

(6)

e) Using (1) from (b):

$$h_{20} = 1 = A\left(\frac{3}{4}\right)^{20} + B\left(\frac{4}{3}\right)^{20}$$

$$= A\left\{\left(\frac{3}{4}\right)^{20} + \frac{3}{4}\cdot\left(\frac{4}{3}\right)^{20}\right\} \quad \text{using } B = \tfrac{3}{4}A \text{ in (d)}$$

$$\therefore \underline{\underline{A = \frac{1}{\left(\frac{3}{4}\right)^{20} + \left(\frac{4}{3}\right)^{19}}}}$$

using $B = \frac{3}{4}A$

So solution (using (c)) is:

$$\underline{\underline{h_x = \frac{1}{\left(\frac{3}{4}\right)^{20} + \left(\frac{4}{3}\right)^{19}} \cdot \left\{\left(\frac{3}{4}\right)^x + \left(\frac{4}{3}\right)^{x-1}\right\}}} \quad \text{as stated.}$$

f) $\mathbb{P}(\text{end in Lost} \mid \text{start at } 1) = 1 - h_1$

$$= 1 - \frac{1}{\left(\frac{3}{4}\right)^{20} + \left(\frac{4}{3}\right)^{19}} \cdot \left\{\frac{3}{4} + \left(\frac{4}{3}\right)^0\right\}$$

$$\underline{\underline{= 0.9926}}$$

g) Yes, T is defective, because $h_1 < 1$ so we might never reach state 20 from state 1. So we can have $T = \infty$ with positive probability.

5a) From the diagram,

$$T = \begin{cases} 1 & \text{with probability } 1/3 \\ 1 + T' + T'' & \text{w.p. } 2/3 \end{cases} \quad \text{where } T' \sim T'' \sim T$$

and T', T'' are independent

$$\text{So} \quad s^T = \begin{cases} s & \text{w.p. } 1/3 \\ s^{1+T'+T''} & \text{w.p. } 2/3 \end{cases}$$

5a cont.) So

$$H(s) = \mathbb{E}(s^T) = \tfrac{1}{3}s + \tfrac{2}{3}s\,\mathbb{E}(s^{T'})\,\mathbb{E}(s^{T''}) \quad \text{because } T', T'' \text{ are indept}$$

$$= \tfrac{1}{3}s + \tfrac{2}{3}s\,H(s)^2 \quad \text{because } T' \sim T'' \sim T$$

$$\therefore \quad 2s\,H(s)^2 - 3H(s) + s = 0$$

$$\therefore \quad \underline{\underline{H(s) = \frac{3 \pm \sqrt{9 - 4 * 2s^2}}{4s} = \frac{3 \pm \sqrt{9-8s^2}}{4s}}}$$

as stated.

5b) $H(s)$ should be continuous as $s \to 0$, and $H(0) = \mathbb{P}(T=0) = 0$.
Using the $+$ root: $\lim_{s\to 0} H(s) = \lim_{s \to 0} \frac{6}{4s} \neq 0$.

So the $+$ root is not possible.

c) T is defective $\iff H(1) < 1$.
Now $H(1) = \frac{3 - \sqrt{9-8}}{4} = 0.5$.

So $H(1) < 1$, $\therefore$ $\underline{\underline{T \text{ is defective}}}$.

d) $p_1 = \mathbb{P}(T = \infty) = 1 - H(1) = 1 - \frac{1}{2} = \underline{\underline{\tfrac{1}{2}}}$ as stated

e) Simplest method is to use PGFs:

Let W = #steps to reach state n, starting from 0.

Let $G_W(s) = \mathbb{E}(s^W)$.

Now $W = T_1 + T_2 + \ldots + T_n$ where $T_1, \ldots, T_n$ are indept, $\sim T$.

5e cont.) So $G_W(s) = \mathbb{E}(s^{T_1 + \cdots + T_n})$

$$= \{H(s)\}^n \quad \text{because } T_1 \sim \ldots \sim T_n \sim T, \text{ indept}$$

So

$$\mathbb{P}(\text{never reach } n) = \mathbb{P}(W = \infty)$$

$$= 1 - G_W(1)$$

$$= 1 - H(1)^n$$

$$\therefore \quad \underline{\underline{\mathbb{P}(\text{never reach } n) = 1 - \left(\tfrac{1}{2}\right)^n}} \quad \text{by (c)}$$

as stated.
Valid for $n = 1, 2, \ldots$

Alternative proof by induction:

Base case: $n=1$ $\quad LHS = p_1 = 1 - \left(\frac{1}{2}\right)^1 = RHS$ by part (d).

Suppose $\circledast$ is true for $n = x$, so can assume

$$p_x = 1 - \left(\tfrac{1}{2}\right)^x \quad ⓐ \text{ allowed}$$

R.T.P. $\quad p_{x+1} = 1 - \left(\frac{1}{2}\right)^{x+1} \quad (**)$

$$\text{LHS of } (**) = p_{x+1} = \mathbb{P}(\text{never reach } x+1 \mid \text{start at } 0)$$

$$= \mathbb{P}(\text{never reach } x \mid \text{start at } 0)$$

$$+ \mathbb{P}(\text{reach } x \mid \text{start at } 0)\,\mathbb{P}(\text{never get from } x \to x+1)$$

$$= p_x + (1 - p_x)\,p_1 \quad \text{using definitions}$$

$$= 1 - \left(\tfrac{1}{2}\right)^x + \left(\tfrac{1}{2}\right)^x \cdot \tfrac{1}{2} \quad \text{using allowed ⓐ}$$

$$= 1 - \left(\tfrac{1}{2}\right)^x \left\{1 - \tfrac{1}{2}\right\} \quad \text{rearranging}$$

$$= 1 - \left(\tfrac{1}{2}\right)^{x+1} = \underline{\text{RHS of } (**)}. \quad \boxed{\text{Goto ✡}}$$

✡

$\therefore$ $\circledast$ is true for $n=1$, and true for $x \Rightarrow$ true for $x+1$, So $\circledast$ is true for all $n = 1, 2, \ldots$

6a) $m_1 = 1 + \frac{3}{4} m_1 + \frac{1}{4} m_2 \Rightarrow m_1 = 4 + m_2$ (a)

$m_2 = 1 + \frac{1}{4} m_1 + \frac{3}{4} * 0 \Rightarrow m_2 = 1 + \frac{1}{4} m_1$ (b)

$m_3 = 0$ by inspection.

(b) in (a) $\Rightarrow$ $m_1 = 4 + 1 + \frac{1}{4} m_1$

$\frac{3}{4} m_1 = 5$

$\therefore \underline{m_1 = \frac{20}{3}}$, so (b) $\Rightarrow$ $\underline{m_2 = \frac{8}{3}}$

Thus $\underset{\sim}{m} = \left(\frac{20}{3}, \frac{8}{3}, 0\right)$ as stated.

b) Consider $T = \begin{cases} 1 + T' & \text{w.p. } 3/4 \\ 1 + U' & \text{w.p. } 1/4 \end{cases}$ $\quad T' \sim T$, $U' \sim U$

and $U = \begin{cases} 1 + T'' & \text{w.p. } 1/4 \\ 1 & \text{w.p. } 3/4 \end{cases}$ $\quad T'' \sim T$

Thus $T^2 = \begin{cases} 1 + 2T' + (T')^2 & \text{w.p. } 3/4 \\ 1 + 2U' + (U')^2 & \text{w.p. } 1/4 \end{cases}$ (1)

and $U^2 = \begin{cases} 1 + 2T'' + (T'')^2 & \text{w.p. } 1/4 \\ 1 & \text{w.p. } 3/4 \end{cases}$ (2)

Note from (a) that $\underline{\mathbb{E}(T) = \frac{20}{3}}$ and $\underline{\mathbb{E}(U) = \frac{8}{3}}$. (3)

(1) and (3) $\Rightarrow$

($T' \sim T$ and $U' \sim U$)

$$\mathbb{E}(T^2) = \frac{3}{4}\{1 + 2\mathbb{E}T + \mathbb{E}(T^2)\} + \frac{1}{4}\{1 + 2\mathbb{E}U + \mathbb{E}(U^2)\}$$

6b cont.) So $\mathbb{E}(T^2) = 4 + 6 * \frac{20}{3} + 2 * \frac{8}{3} + \mathbb{E}(U^2)$ rearranging

i.e. $\underline{\mathbb{E}(T^2) = \frac{148}{3} + \mathbb{E}(U^2)}$ (a)

Also (2), (3) $\Rightarrow$

$\mathbb{E}(U^2) = 1 + \frac{2}{4}\mathbb{E}(T) + \frac{1}{4}\mathbb{E}(T^2)$ because $T'' \sim T$

$= 1 + \frac{1}{2} * \frac{20}{3} + \frac{1}{4}\mathbb{E}(T^2)$

$\underline{\mathbb{E}(U^2) = \frac{13}{3} + \frac{1}{4}\mathbb{E}(T^2)}$ (b)

Substitute (b) in (a) $\Rightarrow$ $\mathbb{E}(T^2) = \frac{148}{3} + \frac{13}{3} + \frac{1}{4}\mathbb{E}(T^2)$

$\therefore \underline{\underline{\mathbb{E}(T^2) = \frac{644}{9} = 71.56}}$

Thus $Var(T) = \mathbb{E}(T^2) - (\mathbb{E}T)^2$

$= \frac{644}{9} - \left(\frac{20}{3}\right)^2$

$\therefore \underline{\underline{Var(T) = \frac{244}{9} = 27.11}}$

7a) Define $m_x = \mathbb{E}(\#\text{tosses to finish} \mid \text{start on square } x)$ for $x = 1, 2, \ldots, 6$.

Then $m_1 = 1 + \frac{1}{2} m_4 + \frac{1}{2} m_3$ (a)

land on Square 2 & climb ladder to Square 4

$m_3 = 1 + \frac{1}{2} m_4 + \frac{1}{2} m_3$ (b)

land on Sq 5 & slip back to Sq 3

$m_4 = 1 + \frac{1}{2} m_3 + \frac{1}{2} * 0$ (c)

7a cont.) Subst (c) in (b):

$$\tfrac{1}{2} m_3 = 1 + \tfrac{1}{2} + \tfrac{1}{4} m_3$$

$$\Rightarrow \underline{m_3 = 6 \text{ tosses}}$$

In (c) $\Rightarrow$ $\underline{m_4 = 4 \text{ tosses}}$

In (a) $\Rightarrow$ $m_1 = 1 + \frac{4}{2} + \frac{6}{2}$

$$\Rightarrow \underline{\underline{m_1 = \mathbb{E}(T) = 6 \text{ tosses}}}.$$

7b) R.T.P. $\mathbb{E}(X) = \sum_{x=1}^{\infty} \mathbb{P}(X \geq x)$. (*)

RHS of (*) $= \sum_{x=1}^{\infty} \mathbb{P}(X \geq x)$

$= \mathbb{P}(X \geq 1) + \mathbb{P}(X \geq 2) + \mathbb{P}(X \geq 3) + \dots$

$[\mathbb{P}(X \geq 1)] \rightarrow$ $= \mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3) + \mathbb{P}(X=4) + \dots$

$[\mathbb{P}(X \geq 2)] \rightarrow$ $+ \mathbb{P}(X=2) + \mathbb{P}(X=3) + \mathbb{P}(X=4) + \dots$

$[\mathbb{P}(X \geq 3)] \rightarrow$ $+ \mathbb{P}(X=3) + \mathbb{P}(X=4) + \dots$

$[\mathbb{P}(X \geq 4)] \rightarrow$ $+ \mathbb{P}(X=4) + \dots$

$+ \dots$

$= 1 * \mathbb{P}(X=1) + 2 * \mathbb{P}(X=2) + 3 * \mathbb{P}(X=3) + \dots$

$= \sum_{x=1}^{\infty} x\, \mathbb{P}(X=x)$

$= \mathbb{E}(X)$ by definition

$= \underline{\underline{\text{LHS of } (*)}}.$

7b cont.)

Alternative argument (more formal):

RHS of (*) $= \sum_{x=1}^{\infty} \mathbb{P}(X \geq x)$

$= \sum_{x=1}^{\infty} \sum_{x} \mathbb{P}(X=y)$

$= \sum_{y=1}^{\infty} \sum_{x \leq y} \mathbb{P}(X=y)$ exchanging order of summation

$= \sum_{y=1}^{\infty} \mathbb{P}(X=y) \sum_{x \leq y} 1$

$= \sum_{y=1}^{\infty} \mathbb{P}(X=y) * y$

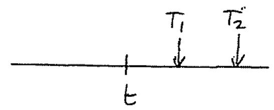
$= \underline{\text{LHS of } (*)}$

7c) The game finishes when the first player reaches Square 6,

so $X = \min(T_1, T_2)$, where $T_1 \sim T_2 \sim T$ and T_1, T_2 are independent.

(T_1 = #tosses for Player 1 to finish, T_2 = #tosses for Pl 2 to finish)

So $\mathbb{P}(X \geq t) = \mathbb{P}(\min(T_1, T_2) \geq t)$



$= \mathbb{P}(T_1 \geq t \text{ AND } T_2 \geq t)$

$= \mathbb{P}(T_1 \geq t)\mathbb{P}(T_2 \geq t)$ by independence of T_1, T_2

$\therefore \underline{\mathbb{P}(X \geq t) = [\mathbb{P}(T \geq t)]^2}$ because $T_1 \sim T_2 \sim T$.

$\therefore$ By (b), $\underline{\underline{\mathbb{E}X = \sum_{t=1}^{\infty} \mathbb{P}(X \geq t) = \sum_{t=1}^{\infty} \{\mathbb{P}(T \geq t)\}^2}}$ as stated