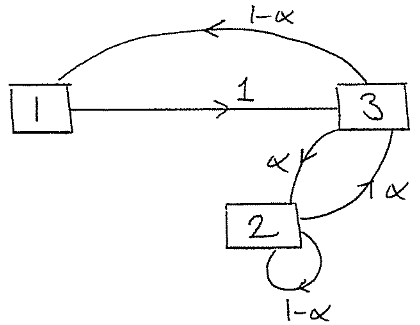
STATS 325 — Exam Solutions — Semester 2, 2010

1a)



b) $\mathbb{P}(X_1=2,\ X_2=3,\ X_3=1) = \mathbb{P}(X_1=2)\,p_{23}\,p_{31}$

$[\mathbb{P}(X_1=2)=\frac{1}{3}$ from Question$]$ $\quad = \frac{1}{3} * \alpha * (1-\alpha)$

$$= \frac{\alpha(1-\alpha)}{3}$$

c) Need to confirm that $\pi_1 + \pi_2 + \pi_3 = 1$ ①

and $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$ ②

$$\text{LHS of ①} = \frac{1}{3-\alpha}\{1-\alpha+1+1\}$$
$$= \frac{3-\alpha}{3-\alpha}$$
$$= 1$$
$$= \text{RHS of ①} \qquad \text{So ① holds.}$$

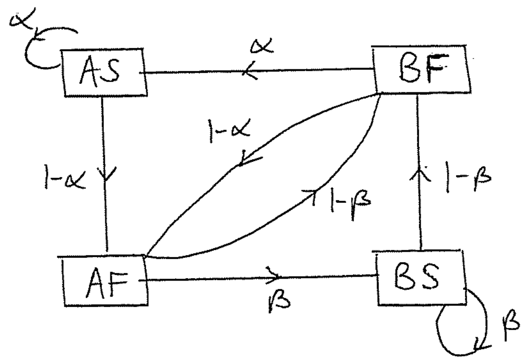
$$\text{LHS of ②} = \underset{\sim}{\pi}^T P = \left(\frac{1-\alpha}{3-\alpha},\ \frac{1-\alpha+\alpha}{3-\alpha},\ \frac{1-\alpha+\alpha}{3-\alpha}\right)$$
$$= \left(\frac{1-\alpha}{3-\alpha},\ \frac{1}{3-\alpha},\ \frac{1}{3-\alpha}\right)$$
$$= \underset{\sim}{\pi}^T = \text{RHS of ②}. \qquad \text{So ② holds}$$

1d) The chain is irreducible.
The loop on state 2 immediately implies it is aperiodic (period (2) = period (all) = 1).
So the chain does converge to $\underset{\sim}{\pi}^T$ as $t \to \infty$.

2a) Matrix:

$$P = \begin{array}{c} \\ AS \\ AF \\ BS \\ BF \end{array} \begin{array}{c} \begin{array}{cccc} AS & AF & BS & BF \end{array} \\ \begin{pmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & \beta & 1-\beta \\ 0 & 0 & \beta & 1-\beta \\ \alpha & 1-\alpha & 0 & 0 \end{pmatrix} \end{array}$$

Diagram:



b) Need $\underset{\sim}{\pi}^T = (\pi_1, \pi_2, \pi_3, \pi_4)$ such that $\begin{cases} \underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T \\ \pi_1+\pi_2+\pi_3+\pi_4 = 1. \end{cases}$

Thus (multiplying out $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$):

$$\alpha\pi_1 + \alpha\pi_4 = \pi_1 \implies \pi_1 = \frac{\alpha}{1-\alpha}\pi_4 \quad ①$$

$$(1-\alpha)\pi_1 + (1-\alpha)\pi_4 = \pi_2 \implies (\alpha + 1-\alpha)\pi_4 = \pi_2 \quad \text{using ①}$$
$$\implies \pi_2 = \pi_4 \quad ②$$

2b) cont. (3) $\beta \pi_2 + \beta \pi_3 = \pi_3 \implies \pi_3 = \frac{\beta}{1-\beta}\pi_4$ using (2)

[Ignore last matrix eqn: will not be linearly indept]

Finally $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

$$\implies \pi_4 \left\{ \frac{\alpha}{1-\alpha} + 1 + \frac{\beta}{1-\beta} + 1 \right\} = 1 \quad \text{using (1),(2),(3)}$$

$$\pi_4 \left\{ \frac{\alpha - \alpha\beta + 2(1-\alpha)(1-\beta) + \beta - \alpha\beta}{(1-\alpha)(1-\beta)} \right\} = 1$$

$$\therefore \pi_4 = \frac{(1-\alpha)(1-\beta)}{\alpha + \beta - 2\alpha\beta + 2 - 2(\alpha+\beta) + 2\alpha\beta}$$

$$\pi_4 = \frac{(1-\alpha)(1-\beta)}{2-\alpha-\beta}$$

Using (1), (2), (3):

$$\underline{\pi}^T = \frac{1}{2-\alpha-\beta}\left(\alpha(1-\beta),\ (1-\alpha)(1-\beta),\ (1-\alpha)\beta,\ (1-\alpha)(1-\beta)\right)$$

2c) $\mathbb{P}(\text{success}) = \pi_1 + \pi_3$ (the two success states: AS and BS)

$$= \frac{\alpha(1-\beta) + \beta(1-\alpha)}{2-\alpha-\beta}$$

$$\mathbb{P}(\text{success}) = \frac{\alpha + \beta - 2\alpha\beta}{2-\alpha-\beta} \quad \text{as stated.}$$

3a) $G(s) = \frac{1}{10}(2 + 5s + 3s^2) \implies \frac{1}{0!}G(0) = \frac{2}{10} = 0.2$

$G'(s) = \frac{1}{10}(5 + 6s) \implies \frac{1}{1!}G'(0) = \frac{5}{10} = 0.5$

$G''(s) = \frac{6}{10} \implies \frac{1}{2!}G''(0) = \frac{1}{2}\cdot\frac{6}{10} = 0.3.$

Clearly $G^{(r)}(s) = 0$ for all $r > 2$.

Thus, using the Attachment, we have $\mathbb{P}(Y=y) = \frac{1}{y!}G^{(y)}(0)$,

so we get

y	0	1	2
$\mathbb{P}(Y=y)$	0.2	0.5	0.3

as stated.

b) $\mathbb{P}(Z_2 = 0) = G(G(0))$

$= G(0.2)$

$= \frac{1}{10}(2 + 5*0.2 + 3*0.2^2)$

$= 0.312$ as stated.

c) $\mathbb{P}(Z_2=0) = \mathbb{P}(Z_2=0 \mid Z_1=0)\mathbb{P}(Z_1=0)$

$+ \mathbb{P}(Z_2=0 \mid Z_1=1)\mathbb{P}(Z_1=1)$

$+ \mathbb{P}(Z_2=0 \mid Z_1=2)\mathbb{P}(Z_1=2)$

Note that $Z_1 \sim Y$ so its possible values are 0, 1, 2.

Now $\mathbb{P}(Z_2=0 \mid Z_1=0) = 1$

$\mathbb{P}(Z_2=0 \mid Z_1=1) = \mathbb{P}(Y=0) = 0.2$ (1 indiv has 0 offspring)

$\mathbb{P}(Z_2=0 \mid Z_1=2) = \mathbb{P}(Y_1=0)\mathbb{P}(Y_2=0) = (0.2)^2 = 0.04$

(2 indivs each have 0 offspring)

3c cont.) So $\mathbb{P}(Z_2=0) = 1*0.2 + 0.2*0.5 + 0.04*0.3$

$= 0.312$ as before.

3d) γ solves $G(s) = s$ (note that $\mathbb{E}(Y) > 1$).

$$\therefore \quad \frac{1}{10}\left(2 + 5s + 3s^2\right) = s$$

$$3s^2 - 5s + 2 = 0$$

$$(s-1)(3s-2) = 0 \qquad \text{(know } (s-1) \text{ is a factor)}$$

$$\therefore \ s = 1 \quad \text{or} \quad s = \frac{2}{3}.$$

γ is minimal solution ≥ 0, so $\gamma = \frac{2}{3}$.

e) If $Z_2 = 3$ then $\mathbb{P}(\text{extinction}) = \gamma^3 = 0.296$.

f) T is defective, because extinction might never occur, in which case $T = \infty$.

$\mathbb{P}(T=\infty) = \mathbb{P}(\text{extinction never occurs}) = 1 - \gamma = \frac{1}{3}$.

g) $\mathbb{P}(T > 2) = 1 - \mathbb{P}(Z_2 = 0)$ by definitions: still indivs alive at time 2.

$= 1 - 0.312$ from (b)

$\mathbb{P}(T>2) = 0.688$

h) If family size $X \sim Y + 3$, it is impossible to have 0 offspring:

x	3	4	5
$\mathbb{P}(X=x)$	0.2	0.5	0.3

So $\mathbb{P}(\text{extinction}) = 0$.

4a) $\mathbb{P}(X_3 = 0 \mid X_0 = 3) = \left(\frac{1}{4}\right)^3 = 0.016$

(Only way of getting $X_3 = 0$ from $X_0 = 3$ is $3 \to 2 \to 1 \to 0$

b) First-step analysis:

$$h_x = \frac{1}{4} h_{x-1} + \frac{3}{4} h_{x+1} \quad \text{for } x = 1, 2, \ldots$$

So $\quad 3h_{x+1} - 4h_x + h_{x-1} = 0 \quad (*) \quad (x = 1, 2, \ldots)$

Clearly, $h_0 = 1$.

c) $3t^2 - 4t + 1 = 0$

$(3t-1)(t-1) = 0$

$\therefore \ t_1 = \frac{1}{3}, \quad t_2 = 1.$

So $\quad h_x = A\left(\frac{1}{3}\right)^x + B(1^x) = \frac{A}{3^x} + B \quad (x = 0, 1, \ldots)$

Boundary: $h_0 = A + B = 1$

$\Rightarrow B = 1 - A$.

So $\quad h_x = A\left\{\frac{1}{3^x} - 1\right\} + 1$

$\therefore \quad h_x = 1 - A\left\{1 - \left(\frac{1}{3}\right)^x\right\} \quad \text{for } x = 0, 1, 2, \ldots$

as stated.

4d) $h_x = 1 - A\left\{ \underbrace{1-\left(\tfrac{1}{3}\right)^x}_{0 \le \text{this} \le 1 \text{ for all } x=0,1,2,\ldots} \right\}$ for $x = 0,1,2,\ldots$

Minimum allowed solution occurs at maximum allowed A.

Consider $x \to \infty$, then $h_x \to 1 - A \geq 0$ ← because h_x is a probability

So $A \leq 1$.

Thus maximum allowed A is $\underline{\underline{A = 1}}$.

Hence

$$\underline{\underline{h_x = \left(\tfrac{1}{3}\right)^x \quad \text{for all } x = 0,1,2,\ldots}}$$

5a)
$$\begin{aligned} G_X(t) &= \mathbb{E}(t^X) \\ &= \sum_{x=0}^{\infty} t^x \, \mathbb{P}(X=x) \\ &= \sum_{x=0}^{\infty} t^x \alpha (1-\alpha)^x \\ &= \alpha \sum_{x=0}^{\infty} \{(1-\alpha)t\}^x \end{aligned}$$

$$\therefore \underline{\underline{G_X(t) = \frac{\alpha}{1-(1-\alpha)t}}} \quad \text{(Geometric series).}$$

Valid for $|t| < \frac{1}{1-\alpha}$.

b)
$$\begin{aligned} G_Y(s) = \mathbb{E}(s^Y) &= \sum_{y=0}^{n} s^y \, \mathbb{P}(Y=y) \\ &= \sum_{y=0}^{n} s^y \binom{n}{y} p^y (1-p)^{n-y} \\ &= \sum_{y=0}^{n} \binom{n}{y} (ps)^y (1-p)^{n-y} \end{aligned}$$

5b cont.) So $\underline{\underline{G_Y(s) = (ps+1-p)^n}}$ by the Binomial Thm.

Valid for all $s \in \mathbb{R}$.

5c) $X \sim \text{Geom}(\alpha)$ so $\mathbb{E}(t^X) = \dfrac{\alpha}{1-(1-\alpha)t}$. (a)

$[Y|X] \sim \text{Bin}(X, p)$ so $\mathbb{E}(s^Y | X) = (ps+1-p)^X$ (b)

So

$$\begin{aligned} G_Y(s) &= \mathbb{E}(s^Y) \\ &= \mathbb{E}_X\{\mathbb{E}(s^Y|X)\} && \text{law of total expectation} \\ &= \mathbb{E}_X\{(ps+1-p)^X\} && \text{by (b)} \\ &= G_X(ps+1-p) && \text{by definition of } G_X \\ &= \frac{\alpha}{1-(1-\alpha)(ps+1-p)} && \text{by (a) with } t = ps+1-p \\ &= \frac{\alpha}{\cancel{1} - (1-\alpha)ps - \cancel{1} + \alpha + p - \alpha p} \end{aligned}$$

$$\therefore \underline{\underline{G_Y(s) = \frac{\alpha}{\alpha + p(1-\alpha) - (1-\alpha)ps}}} \quad \text{as stated.}$$

Divide by $\{\alpha + p(1-\alpha)\}$ to get a 1 on the denominator:

$$G_Y(s) = \frac{\alpha / \{\alpha+p(1-\alpha)\}}{1 - p(1-\alpha)s / \{\alpha+p(1-\alpha)\}} = \frac{\beta}{1-(1-\beta)s}.$$

for some β.

So $\underline{\underline{Y \sim \text{Geometric}(\beta) \sim \text{Geometric}\left(\frac{\alpha}{\alpha+p(1-\alpha)}\right)}}$.

6a) $U = \begin{cases} 1 & \text{w.p. } 1/4 \\ 1+U' & \text{w.p. } 1/2 \\ 1+U''+U''' & \text{w.p. } 1/4 \end{cases}$ where $U' \sim U'' \sim U''' \sim U$ and U'', U''' indept.

So $H_u(s) = \mathbb{E}(s^u) = \frac{1}{4}s + \frac{1}{2}s\underbrace{H_u(s)}_{\substack{\text{because}\\ U'\sim U}} + \frac{1}{4}s\underbrace{H_u(s)^2}_{\substack{\text{because } U''\sim U'''\\ \sim U, \text{ and}\\ U'', U''' \text{ indept.}}}$

So $s H_u(s)^2 + (2s-4)H_u(s) + s = 0$

$$\therefore \quad H_u(s) = \frac{4-2s \pm \sqrt{\cancel{4s^2} - 16s + 16 - \cancel{4s^2}}}{2s}$$

$$= \frac{2(2-s) \pm \sqrt{16}\sqrt{1-s}}{2s}$$

$$H_u(s) = \frac{2-s \pm 2\sqrt{1-s}}{s} \quad \text{as stated.}$$

b) We know that $H_u(0) = \mathbb{P}(U=0) = 0$.

Consider the (+) root:

$$\lim_{s\to 0}\left\{\frac{2-s+2\sqrt{1-s}}{s}\right\} = \lim_{s\to 0}\left(\frac{4}{s}\right) \neq 0 \quad \text{(does not exist.)}$$

So the (+) root is not correct.

c) U is defective $\Leftrightarrow H_u(1) < 1$

Now $H_u(1) = \frac{2-1-2\sqrt{1-1}}{1} = 1$.

So U is not defective.

6d) Clearly $V \sim U$ by symmetry. So $H_v(s) \equiv H_u(s)$.

e) $T = \begin{cases} 1+V & \text{w.p. } 1/4 & \text{(go } 0\to 1 \text{ on first step, need to get back } 1\to 0) \\ 1 & \text{w.p. } 1/2 & \text{(return immediately)} \\ 1+U & \text{w.p. } 1/4 & \text{(go } 0\to -1 \text{ on first step, need to get back } -1\to 0) \end{cases}$

So $G(s) = \mathbb{E}(s^T) = \frac{1}{4}sH_v(s) + \frac{1}{2}s + \frac{1}{4}sH_u(s)$

$= \frac{1}{2}s + \frac{1}{2}sH_u(s)$ because $H_v = H_u$

$= \frac{1}{2}s\left\{1 + \frac{2-s-2\sqrt{1-s}}{s}\right\}$ by (c)

$= \frac{1}{2}s\left\{\frac{2-2\sqrt{1-s}}{s}\right\}$

$\therefore \quad G(s) = 1 - \sqrt{1-s}$ as stated.

f) $\mathbb{E}(T) = G'(1)$.

Now $G(s) = 1 - \{1-s\}^{1/2}$

So $G'(s) = +\frac{1}{2}(1-s)^{-1/2} = \frac{1}{2\sqrt{1-s}}$

$\therefore \quad \mathbb{E}T = G'(1) = \infty$.

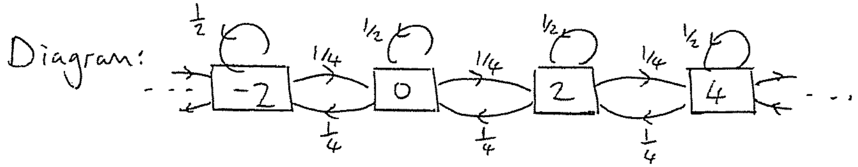
g) Consider $X_t = Y_t - Z_t$. Given that $X_0 = 0$.

Need the transition diagram of X_t.

$Y_{t+1} - Y_t$	$Z_{t+1} - Z_t$	$X_{t+1} - X_t$	Prob.
1	1	0	1/4
1	-1	2	1/4
-1	1	-2	1/4
-1	-1	0	1/4

6g cont.) So
$$X_{t+1} = \begin{cases} X_t + 2 & \text{w.p. } 1/4 \\ X_t & \text{w.p. } 1/2 \\ X_t - 2 & \text{w.p. } 1/4 \end{cases}$$

Diagram:



Thus it is clear that $W \sim T$, because the new diagram is identical to the diagram in part (a) except for the labels on the states, which don't matter.

So $\underline{\underline{\mathbb{E}(s^W) = \mathbb{E}(s^T) = 1 - \sqrt{1-s}}}$

Now $G(1) = 1 - \sqrt{0} = 1.$

So W is not defective,
so the two random walks <u>will</u> certainly meet again.

7a) $\mathbb{P}(\text{visits state } 1) = \mathbb{P}(\text{hits state 1 before state 2}).$

Define $h_x = \mathbb{P}(\text{hits state 1 before state 2} \mid \text{start at state } x).$

We want h_5.

$$h_5 = p*1 + q h_4 \quad ①$$
$$h_4 = p h_5 + q h_3 \quad ②$$
$$h_3 = p h_4 + q*0 \quad (h_2 = 0) \quad ③$$

②,③ $\Rightarrow h_4 = p h_5 + pq h_4, \quad h_4 = \frac{p}{1-pq} h_5$

In ①: $h_5 = p + \frac{pq}{1-pq} h_5$

7a cont.) i) So $\underline{\underline{h_5 = \frac{p(1-pq)}{1-2pq}}}$ prob. visits state 1.

ii) Similarly, define $v_x = \mathbb{P}(\text{hit state 3 before state 2} \mid \text{start at state } x)$

Then
$$v_5 = p v_1 + q v_4 \quad ①$$
$$v_4 = q + p v_5 \quad (v_3 = 1) \quad ②$$
$$v_1 = p*0 + q v_5 \quad ③$$

②,③ in ① $\Rightarrow v_5 = 2pq v_5 + q^2$

So $\underline{\underline{v_5 = \frac{q^2}{1-2pq}}}$. prob. visits state 3.

When $p = q = \frac{1}{2}$, $\underline{\underline{\mathbb{P}(\text{visits state } 1) = \frac{\frac{1}{2}(1-\frac{1}{4})}{1-\frac{1}{2}} = \frac{3}{4}}}$

$\underline{\underline{\mathbb{P}(\text{visits state } 3) = \frac{1/4}{1-1/2} = \frac{1}{2}}}$.

7b) Define events $V_1 = \{\text{visits state } 1\}$, $V_3 = \{\text{visits state } 3\}$
and $V_{all} = \{\text{visits all states}\}$.

By inspection, $V_{all} = V_1 \cap V_3$
because 5 & 2 are start and end, and visiting 4 is implied by V_3: we can't visit 3 from 5 without going through 4.

So $\mathbb{P}(V_{all}) = \mathbb{P}(V_1 \cap V_3)$

Answer $= \frac{3}{4} + \frac{1}{2} - 1 = \frac{1}{4}$
when $p = q = \frac{1}{2}$.

Now it's clear that $\mathbb{P}(V_1 \cup V_3) = 1$: we must visit either 1 or 3

So $\mathbb{P}(V_1 \cup V_3) = 1 = \mathbb{P}(V_1) + \mathbb{P}(V_3) - \mathbb{P}(V_1 \cap V_3)$

$\therefore \underline{\underline{\mathbb{P}(V_{all}) = \mathbb{P}(V_1 \cap V_3) = \frac{p(1-pq) + q^2}{1-2pq} - 1.}}$