Stats 325 Exam Solutions Semester 2, 2011

1a) Transition matrix:
$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix} \end{array}$$

b) We require $\underset{\sim}{\pi}$ such that $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$ and $\pi_1 + \pi_2 = 1$.

Equations:
$$\alpha \pi_1 + (1-\beta)\pi_2 = \pi_1 \quad ①$$
$$\pi_1 + \pi_2 = 1 \quad ②$$

② in ① $\Rightarrow$ $\alpha\pi_1 + (1-\beta)(1-\pi_1) = \pi_1$
$$1-\beta = \pi_1(1-\alpha+1-\beta)$$
$$\therefore \quad \pi_1 = \frac{1-\beta}{2-\alpha-\beta}$$

Giving overall, $\underset{\sim}{\pi}^T = (\pi_1, \pi_2) = \left(\frac{1-\beta}{2-\alpha-\beta}, \frac{1-\alpha}{2-\alpha-\beta}\right)$.

c) The chain is irreducible and aperiodic, and an equilibrium distribution exists, so yes, the chain does converge to $\underset{\sim}{\pi}$ as $t \to \infty$.

d) $\mathbb{P}(\text{success}) = \mathbb{P}(\text{success} \mid A)\mathbb{P}(A) + \mathbb{P}(\text{success} \mid B)\mathbb{P}(B)$

$= \alpha\pi_1 + \beta\pi_2$ because the chain converges to $\underset{\sim}{\pi}$ as $t \to \infty$

$$= \frac{\alpha(1-\beta) + \beta(1-\alpha)}{2-\alpha-\beta}$$

$\therefore$ $\mathbb{P}(\text{success}) = \dfrac{\alpha+\beta-2\alpha\beta}{2-\alpha-\beta}$ long term, as stated.

1e) Define $p_T = \mathbb{P}(\text{success})$ under two-armed bandit strategy (long term).

Alternative strategy applying A to all patients has long-run success probability $= p_A = \alpha$.

Consider
$$p_A - p_T = \frac{\alpha(2-\alpha-\beta) - (\alpha+\beta-2\alpha\beta)}{2-\alpha-\beta}$$
$$= \frac{2\alpha - \alpha^2 - \alpha\beta - \alpha - \beta + 2\alpha\beta}{2-\alpha-\beta}$$
$$= \frac{\alpha - \alpha^2 + \alpha\beta - \beta}{2-\alpha-\beta}$$
$$= \frac{(\alpha-\beta)(1-\alpha)}{2-\alpha-\beta}$$

$\therefore$ $p_A - p_T > 0$ because $\alpha > \beta$ so $\alpha - \beta > 0$
and $1 > \alpha$ so $1 - \alpha > 0$
and $\alpha, \beta < 1$ so $2-\alpha-\beta > 0$.

We use the two-armed bandit strategy because, in practice, we do not know which of the two treatments is better. So we don't know that $\alpha > \beta$.

2a) Transition matrix:
$$\begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} & \begin{pmatrix} q & p & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & \cdots \\ 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & q & 0 & p & \cdots \\ & & & \ddots & \ddots & \end{pmatrix} \end{array}$$

Equilibrium equations: $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$

so $q\pi_0 + q\pi_1 = \pi_0 \Rightarrow q\pi_1 = p\pi_0$ eqn for π_0.

2a cont.) For $k=1,2,3,\ldots$ we have:

$$p\pi_{k-1} + q\pi_{k+1} = \pi_k$$

so $\underline{q\pi_{k+1} - \pi_k + p\pi_{k-1} = 0 \text{ for } k=1,2,\ldots}$ (*)

as stated.

2b) Let $t=1$. LHS(**) $= q - 1 + p$

$= 0$ because $p+q=1$

$=$ RHS(**). So $t_1=1$ is a root of (**)

So (**) factorises as $(t-1)(qt-p)=0$

$$\Rightarrow \underline{t_1 = 1 \quad t_2 = \frac{p}{q}}$$

So general solution to (*) is

$$\underline{\pi_k = A + B\left(\frac{p}{q}\right)^k \quad k=0,1,2,\ldots}$$

2c) Substituting answer to (b) in equation $p\pi_0 = q\pi_1$:

$$p\{A+B\} = q\{A + B.\tfrac{p}{q}\}$$

$$pA + pB = qA + pB$$

$$\Rightarrow pA = qA \quad \text{equivalently} \quad (p-q)A = 0.$$

But $p \neq q$, so we must have $\underline{A=0.}$

So we must have $\underline{\pi_k = B\left(\frac{p}{q}\right)^k \text{ for } k=0,1,2,\ldots}$

as stated.

2d) For equilibrium to exist, we must have

$$\sum_{k=0}^{\infty} \pi_k = 1$$

i.e. $B \sum_{k=0}^{\infty} \left(\frac{p}{q}\right)^k = 1.$

If $p>q$ the sum diverges and there is no solution for B.

If $p<q$ then $B \cdot \frac{1}{1-\frac{p}{q}} = 1$ (geometric series)

$$\Rightarrow \underline{B = 1 - \frac{p}{q}}.$$

So $\underline{\text{equilibrium distn exists} \Leftrightarrow p<q \text{ (equivalently, } p<\tfrac{1}{2})}$

If it does exist, then $\underline{\pi_k = \left(1-\frac{p}{q}\right)\left(\frac{p}{q}\right)^k \text{ for } k=0,1,2,\ldots}$

substituting $B = 1-\frac{p}{q}$ into (c).

2e) The chain is irreducible and aperiodic, so if an equilibrium distribution exists we will have convergence.

So $\underline{\text{the chain converges to equilibrium} \Leftrightarrow p<q \Leftrightarrow p<\tfrac{1}{2}}$.

2f) $p=0.3 < \frac{1}{2}$ so the chain converges.

So Long-run proportion of time in state $0 = \pi_0 = \left(1-\frac{0.3}{0.7}\right)\left(\frac{0.3}{0.7}\right)^0$

$$\therefore \underline{\pi_0 = \frac{4}{7}} \quad (0.571)$$

2g) For $X_0 \sim \text{Geom}\left(\frac{4}{7}\right)$, $\mathbb{P}(X_0=k) = \frac{4}{7}\left(\frac{3}{7}\right)^k \quad (k=0,1,\ldots)$

$= \pi_k$ when $p=0.3$.

So $X_0 \sim \underset{\sim}{\pi}^T$, so by definition of equilibrium, $\underline{X_1 \sim \underset{\sim}{\pi}^T \sim \text{Geom}\left(\frac{4}{7}\right) \text{al}}$

3a) Consider

$$U = \begin{cases} 1 & \text{with prob. } 1/3 \\ 1+U^* & \text{w.p. } 1/3 \quad \text{where } U^* \sim U \\ 1+U'+U'' & \text{w.p. } 1/3 \quad \text{where } U' \sim U'' \sim U \end{cases}$$

and U', U'' independent.

So $H_u(s) = \mathbb{E}(s^U) = \frac{1}{3}s + \frac{1}{3}s\,\mathbb{E}(s^{U^*}) + \frac{1}{3}s\,\mathbb{E}(s^{U'}s^{U''})$

$\therefore \quad H_u(s) = \frac{1}{3}s + \frac{1}{3}s H_u(s) + \frac{1}{3}s\left[H_u(s)\right]^2$

because $U^* \sim U' \sim U'' \sim U$ and U', U'' indep

Rearranging: $s H_u(s)^2 + (s-3)H_u(s) + s = 0$

$\therefore \quad H_u(s) = \dfrac{3-s \pm \sqrt{9-6s+s^2-4s^2}}{2s}$

$\therefore \quad \underline{\underline{H_u(s) = \dfrac{3-s \pm \sqrt{9-6s-3s^2}}{2s}}}$ as stated.

b) We need $\lim_{s\to 0} H_u(s) = \mathbb{P}(U=0) = 0$, because it takes ≥ 1 step to travel $0 \to 1$.

With the $(+)$ root, $\lim_{s\to 0} H_u(s) = \lim_{s \to 0} \frac{6}{s} \neq 0$ (infinity or undefined)

So the $(+)$ root can not be correct.

c) $H_u(1) = \dfrac{3-1-\sqrt{9-6-3}}{2} = \dfrac{2-\sqrt{0}}{2} = 1.$

So $\underline{\underline{U \text{ is not defective.}}}$

d) Clearly $V \sim U$ by symmetry, so $H_V(\cdot) = H_u(\cdot)$.

3e) Consider

$$T = \begin{cases} 1 & \text{w.p. } 1/3 & \text{1st step = loop} \\ 1+U & \text{w.p. } 1/3 & \text{1st step down, needs } U \text{ to ret} \\ 1+V & \text{w.p. } 1/3 & \text{1st step up, needs } V \text{ to return} \end{cases}$$

So $\mathbb{E}(s^T) = G(s) = \frac{1}{3}s + \frac{1}{3}s H_u(s) + \frac{1}{3}s H_V(s)$ $(*)$

But $H_u(s) = H_V(s) = \dfrac{3-s-\sqrt{9-6s-3s^2}}{2s}$

So $(*)$ gives $\quad 3G(s) = s + \cancel{2s}\cdot\dfrac{1}{\cancel{2s}}\cdot\left\{3-s-\sqrt{9-6s-3s^2}\right\}$

$\Rightarrow \quad \underline{\underline{G(s) = 1 - \frac{1}{3}\sqrt{9-6s-3s^2}}}$ as stated.

3f) $\mathbb{P}(\text{never return to state } 5) = \mathbb{P}(T=\infty)$

$= 1 - G(1)$

$= 1 - \left\{1 - \frac{1}{3}\sqrt{9-6-3}\right\}$

$= \underline{\underline{0.}}$

Return is guaranteed.

4a) First-step analysis:

$h_x = \frac{1}{2}h_{x+1} + \frac{1}{2}h_{x-1}$ for $x = 1, 2, \ldots, N-1$.

So $h_{x+1} - 2h_x + h_{x-1} = 0$

$\therefore \quad \underline{\underline{h_{x+1} = 2h_x - h_{x-1} \text{ for } x = 1, 2, \ldots, N-1}}$ as stated.

Clearly, $\underline{\underline{h_0 = 0, \; h_N = 1.}}$

4b) R.T.P. $h_x = x h_1$ for $x = 0, 1, \dots, N$ (*).

We need two base cases because the given information from (a) is

$$h_{x+1} = 2h_x - h_{x-1} \quad (x = 1, 2, \dots, N-1) \quad (a)$$

Base cases: i) $x = 0$:

$$\text{LHS}(*) = h_0 = 0 \text{ by (a)}$$
$$= \text{RHS}(*).$$ So (*) holds for $x = 0$.

ii) $x = 1$: $\text{LHS}(*) = h_1$

$\text{RHS}(*) = 1 * h_1 = \text{LHS}(*)$. So (*) holds for $x = 1$.

General case: Suppose (*) holds for $x = 0, 1, \dots, k$ for some k.

So we can assume:

$$h_k = k h_1 \quad (c)$$
$$h_{k-1} = (k-1) h_1 \quad (d)$$

R.T.P. (*) holds for $x = k+1$,

i.e. R.T.P. $h_{k+1} = (k+1) h_1$ (**)

$$\begin{aligned} \text{LHS}(**) = h_{k+1} &= 2h_k - h_{k-1} \text{ by known info (a), provided } k+1 \le N \\ &= 2kh_1 - (k-1)h_1 \text{ using allowed (c) and (d)} \\ &= (k+1)h_1 \\ &= \text{RHS}(**) \end{aligned}$$

Thus we proved (*) true for $x = 0$ and $x = 1$, and if (*) holds for all $x = 0, 1, \dots, k$ we proved (*) holds for $x = k+1$, provided $k+1 \le N$.

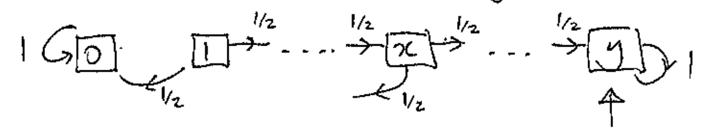
So (*) is proved for all $x = 0, 1, \dots, N$. □

4c) Boundary $h_N = 1 = N h_1$, by (*) and (a).

So $h_1 = \frac{1}{N}$ and therefore $h_x = \frac{x}{N}$ for all $x = 0, \dots, N$.

4d) Consider the event $\{Y \geq y\}$ that the trajectory maximum $\geq y$.

We could draw a new transition diagram:



Attained: trajectory maximum is at least y.

Clearly, $\mathbb{P}(Y \geq y) = \mathbb{P}(\text{walk hits } y \text{ before it hits } 0)$

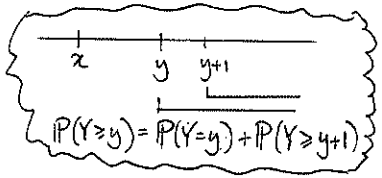
and this is the situation studied in parts (a)–(c) where we substitute $N = y$.

So by (a)–(c) we have

$$\mathbb{P}(Y \geq y \mid \text{start at } x) = \frac{x}{y} \quad \text{for } x = 0, \dots, y$$

i.e. $y = x, \dots, N$.

So $\mathbb{P}(Y = y \mid \text{start at } x) = \mathbb{P}(Y \geq y \mid \text{start } x) - \mathbb{P}(Y \geq y+1 \mid \text{start } x)$



$$= \frac{x}{y} - \frac{x}{y+1}$$

Valid for $y = x, \dots, N-1$.

Clearly, $\mathbb{P}(Y = y \mid \text{start at } x) = \frac{x}{N}$ if $y = N$.

Overall,

$$\mathbb{P}(Y = y) = \begin{cases} 0 & \text{if } y = 0, \dots, x-1 \\ \frac{x}{y} - \frac{x}{y+1} & y = x, \dots, N-1 \\ \frac{x}{N} & \text{if } y = N. \end{cases}$$

5a) $Y \sim \text{Poisson}(\lambda)$. $\mathbb{P}(Y=y) = \frac{\lambda^y}{y!}e^{-\lambda}$ for $y = 0, 1, \dots$

So
$$\begin{aligned} G_Y(s) &= \mathbb{E}(s^Y) \\ &= \sum_{y=0}^{\infty} s^y \cdot \frac{\lambda^y}{y!}e^{-\lambda} \\ &= e^{-\lambda}\sum_{y=0}^{\infty} \frac{(\lambda s)^y}{y!} \quad \text{(exponential power series)} \\ &= e^{-\lambda}e^{\lambda s} \end{aligned}$$

$\underline{\underline{G_Y(s) = e^{\lambda(s-1)}}}$ as stated.

b) Let $T = Y_1 + \dots + Y_n$ where $Y_1, \dots, Y_n \sim \text{Poisson}(\lambda)$, independent.

$$\begin{aligned} G_T(s) = \mathbb{E}(s^T) &= \mathbb{E}(s^{Y_1+\dots+Y_n}) \\ &= \mathbb{E}(s^{Y_1})\dots\mathbb{E}(s^{Y_n}) \quad \text{by indep of } Y_1, \dots, Y_n \\ &= [G_Y(s)]^n \quad \text{because } Y_i \sim Y \sim \text{Poisson}(\lambda) \text{ for all } i \\ &= \left[e^{\lambda(s-1)}\right]^n \end{aligned}$$

$\therefore \underline{\underline{G_T(s) = e^{n\lambda(s-1)}}}$

This is the PGF of the Poisson $(n\lambda)$ distribution.
So $\underline{\underline{T \sim \text{Poisson}(n\lambda)}}$ as stated.

c i) $\mathbb{P}(X_0 = 10, X_1 = 8, X_2 = 12) = \mathbb{P}(X_0 = 10)\mathbb{P}(X_1 = 8 \mid X_0 = 10)\mathbb{P}(X_2 = 12 \mid X_1$

$$\begin{aligned} &= 1 * \mathbb{P}(\text{Poisson}(10) = 8) * \mathbb{P}(\text{Poisson}(8) = 12) \\ &= \frac{10^8}{8!}e^{-10} \cdot \frac{8^{12}}{12!}e^{-8} \\ &= \underline{\underline{0.0054}}. \end{aligned}$$

5c ii) Set up a transition diagram:

$$\begin{tikzcd} \boxed{0} \arrow[loop left, "1"] & \boxed{1} \arrow[l, "\frac{1^0}{0!}e^{-1} = e^{-1}"'] \arrow[loop above, "\frac{1^1}{1!}e^{-1} = e^{-1}"] \arrow[r, "1-2e^{-1}"'] & \boxed{2 \text{ or more}} \\ & & \text{(fail)} \end{tikzcd}$$

Define $p = \mathbb{P}(\text{chain never hits 2 or more})$
First-step analysis:
$$p = e^{-1}p + e^{-1} * 1 + (1-2e^{-1}) * 0$$
$$\Rightarrow p(1-e^{-1}) = e^{-1}$$
$$\underline{\underline{p = \frac{e^{-1}}{1-e^{-1}} = \frac{1}{e-1} = 0.582}}$$
(probability chain never exceeds 1).

5c iii) By (b), $[X_{t+1} \mid X_t] \sim \text{Poisson}(X_t)$
$$\sim \underbrace{\text{Poisson}(1) + \text{Poisson}(1) + \dots + \text{Poisson}(1)}_{X_t \text{ times}}$$

i.e. $[X_{t+1} \mid X_t] \sim Y_1 + Y_2 + \dots + Y_{X_t}$ where each $Y_i \sim \text{Poisson}(1)$ and $Y_1, Y_2, \dots$ are independent.

This process is equivalent to a <u>branching process</u> with family size distribution $\underline{\underline{Y \sim \text{Poisson}(1)}}$. Note that $\mu = \mathbb{E}Y = 1$.

The generation size at time t is $Z_t = X_t$ and the generation size at time $t+1$ is X_{t+1} which is the sum of X_t families:
$$[X_{t+1} \mid X_t] = Y_1 + \dots + Y_{X_t} = \text{sum of offspring of } X_t \text{ paren}$$

Thus $\mathbb{P}(\text{process ever reaches } 0) = \mathbb{P}(\text{eventual extinction})$
$\underline{\underline{= 1 \quad \text{because } \mu = \mathbb{E}Y = 1}}$

Extinction is guaranteed when mean of family size distn is ≤ 1.

6a) For eventual extinction, all k of the individual branching processes started by individuals $1, 2, \dots, k$ must go extinct.

So $\underline{\underline{\mathbb{P}(\text{eventual extinction} \mid \text{start from } k) = \gamma^k}}$ $\quad (k = 0, 1, 2, \dots)$

b) Define $h_k = \mathbb{P}(\text{hit 0 eventually} \mid \text{start from state } k)$

$\therefore \underline{h_k = \mathbb{P}(\text{eventual extinction} \mid \text{start from } k) = \gamma^k}$ by (a).

We want to find $h_1 = \mathbb{P}(\text{extinction} \mid \text{start from } Z_0 = 1)$.

First-step analysis equation:

$$h_1 = \sum_{k=0}^{\infty} \mathbb{P}(Y=k)\, h_k$$

because the transition probabilities starting from 1 individual are given by Y, the offspring size of 1 individual.

Substituting $h_k = \gamma^k$ for all k, we obtain the FSA eqn:

$$\gamma^1 = \sum_{k=0}^{\infty} \gamma^k \, \mathbb{P}(Y=k)$$

$= \mathbb{E}(\gamma^Y)$ by definition of expectation

$= G(\gamma)$ by definition of PGF, G.

Thus the theorem informs us that γ is the minimal non-negative solution of the FSA eqn, $\gamma = G(\gamma)$.

Equivalently, γ is the minimal non-negative solution to the equation $G(s) = s$.

7a) $\mathbb{P}(X_{t+1} = 0 \mid X_t = 5) = \mathbb{P}(U_{t+1} + W_{t+1} = 0 \mid X_t = 5)$

$= \mathbb{P}(U_{t+1} = 0 \mid X_t = 5)\, \mathbb{P}(W_{t+1} = 0 \mid X_t = 5)$

by conditional independence of U_{t+1}, W_{t+1}

$= \underline{\underline{(1-p)^5 (1-a)^{n-5}}}$ because $[U_{t+1} \mid X_t = 5] \sim \text{Bin}(5, p)$, $[W_{t+1} \mid X_t = 5] \sim \text{Bin}(n-5, a)$

$\mathbb{P}(X_{t+1} = 1 \mid X_t = 5) = \mathbb{P}(U_{t+1} = 1 \text{ and } W_{t+1} = 0 \mid X_t = 5)$

$+ \mathbb{P}(U_{t+1} = 0 \text{ and } W_{t+1} = 1 \mid X_t = 5)$ similarly

$= \underline{\underline{5p(1-p)^4 (1-a)^{n-5} + (n-5)a(1-a)^{n-6}(1-p)^5}}$

using similar arguments.

b) $\{X_t\}$ is irreducible, because any number of children could become ill out of $n - X_t$, and any number could remain ill out of X_t, so any number in $0, 1, \dots, n$ is possible for the number of ill at time $t+1$.
$\{X_t\}$ is aperiodic, because there is a loop on every state.
The state space $S = \{0, \dots, n\}$ so an equilibrium distribution exists. So $\{X_t\}$ <u>does</u> converge to equilibrium as $t \to \infty$.

c) $Y \sim \text{Bin}(m, \beta)$ so $\mathbb{P}(Y=y) = \binom{m}{y} \beta^y (1-\beta)^{m-y}$ for $y = 0, \dots,$

$$G_Y(s) = \mathbb{E}(s^Y) = \sum_{y=0}^{m} \binom{m}{y} (\beta s)^y (1-\beta)^{m-y}$$

$\therefore \underline{\underline{G_Y(s) = (\beta s + 1 - \beta)^m}}$ (Binomial Thm) as stated.

d) Consider $X_{t+1} = U_{t+1} + W_{t+1}$, indept given X_t

with $U_{t+1} \mid X_t \sim \text{Bin}(X_t, p)$, $W_{t+1} \mid X_t \sim \text{Bin}(n-X$

7d cont.) So $\mathbb{E}(s^{X_{t+1}} | X_t) = \mathbb{E}(s^{U_{t+1}} | X_t)\, \mathbb{E}(s^{W_{t+1}} | X_t)$ (indep given X_t)

$$= (ps + 1 - p)^{X_t} (as + 1 - a)^{n - X_t} \quad \text{using (c) twic}$$

$$\underline{\underline{\mathbb{E}(s^{X_{t+1}} | X_t) = (as + 1 - a)^n \left\{ \frac{ps + 1 - p}{as + 1 - a} \right\}^{X_t}}} \quad \text{as stated.}$$

7e) Suppose $X_t \sim \text{Bin}(n, \pi)$, so $\mathbb{E}(r^{X_t}) = (\pi r + 1 - \pi)^n$. (*)

So $\mathbb{E}(s^{X_{t+1}}) = \mathbb{E}_{X_t}\{ \mathbb{E}(s^{X_{t+1}} | X_t) \}$ law of total expectation

$$= (as + 1 - a)^n \; \mathbb{E}_{X_t} \left\{ \left(\frac{ps + 1 - p}{as + 1 - a} \right)^{X_t} \right\} \quad \text{using (d)}$$

$$= (as + 1 - a)^n \left\{ \pi \left(\frac{ps + 1 - p}{as + 1 - a} \right) + 1 - \pi \right\}^n \quad \text{using } r = \frac{ps + 1 - p}{as + 1 - a} \text{ in (*)}$$

$$= \{ \pi (ps + 1 - p) + (1 - \pi)(as + 1 - a) \}^n \quad \text{multiplying out}$$

$$= \{ s(\pi p + a - \pi a) + \cancel{\pi} - \pi p + 1 - a - \cancel{\pi} + \pi a \}^n$$

$$\underline{\underline{\mathbb{E}(s^{X_{t+1}}) = \{ s(\pi p + a - \pi a) + 1 - (\pi p + a - \pi a) \}^n}}$$

So if $X_t \sim \text{Bin}(n, \pi)$, then $\underline{\underline{X_{t+1} \sim \text{Bin}(n, \pi p + a - \pi a)}}$.

Thus if we can find π such that $\pi p + a - \pi a = \pi$, then $X_{t+1} \sim X_t$ and $\text{Bin}(n, \pi)$ is an equilibrium distn for X_t.

Need $\pi = \pi p + a - \pi a \Rightarrow \underline{\pi = \frac{a}{1 - p + a}}$.

So an equilibrium distn is $\underline{\underline{\text{Binomial}\left(n, \frac{a}{1 - p + a}\right)}}$.