1) a) $P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$

b) $\mathbb{P}(X_1=2, X_2=2, X_3=1) = \mathbb{P}(X_1=2) \cdot P_{22} P_{21}$

$$= \tfrac{1}{2}(1-\beta)\beta$$

c) Require $\underset{\sim}{\pi} = (\pi_1, \pi_2)$ such that $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$ and $\pi_1 + \pi_2 = 1$:

Eqns $\pi_1(1-\alpha) + \pi_2 \beta = \pi_1$ ①

$\pi_1 + \pi_2 = 1$ ②

Subst ② in ①: $\pi_1(1-\alpha) + (1-\pi_1)\beta = \pi_1$

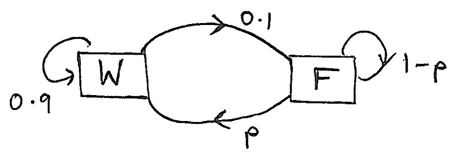
$\pi_1 - (\alpha+\beta)\pi_1 + \beta = \pi_1$

$\Rightarrow \pi_1 = \dfrac{\beta}{\alpha+\beta}$, so $\pi_2 = \dfrac{\alpha}{\alpha+\beta}$.

Thus $\underset{\sim}{\pi}^T = \left(\dfrac{\beta}{\alpha+\beta}, \dfrac{\alpha}{\alpha+\beta} \right)$ Equilibrium distn.

d) Because $\alpha > 0$ and $\beta > 0$, the chain is irreducible.
Because $\alpha < 1$, so $1-\alpha > 0$, so there is a loop on state 1 and thus the chain is aperiodic. $1-\alpha$ $\boxed{1}$

By (c) an equilibrium distribution exists.
So the Markov chain does converge to $\underset{\sim}{\pi}$ for all α, β with $0 < \alpha < 1$ and $0 < \beta \leq 1$.

e) 

1f) The diagram meets the criteria of parts (c) and (d) as long as $0 < p \leq 1$. So it converges to $\underset{\sim}{\pi}^T$ with $\alpha = 0.1$, $\beta = p$:

$$\underset{\sim}{\pi}^T = \left(\frac{p}{0.1+p}, \frac{0.1}{0.1+p} \right) \text{ for any } 0 < p \leq 1.$$

In state W, the machine earns +100 each day.
In state F, it earns $-100p$ each day.
So by convergence to $\underset{\sim}{\pi}$, we have in the long run:

$$\text{Daily earning} = \begin{cases} 100 & \text{in state W } \left(\text{prob } \pi_1 = \frac{p}{0.1+p}\right) \\ -100p & \text{in state F } \left(\text{prob } \pi_2 = \frac{0.1}{0.1+p}\right) \end{cases}$$

$\therefore \mathbb{E}(\text{long-run daily earning}) = 100\pi_1 - 100p\,\pi_2$

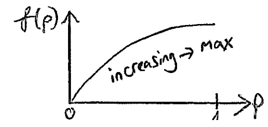
$$= \frac{100p - 100p * 0.1}{0.1 + p}$$

$$\therefore f(p) = \frac{90p}{0.1+p} \quad \text{as stated.}$$

1g) Consider $\dfrac{df}{dp} = \dfrac{(0.1+p)90 - 90p}{(0.1+p)^2}$

$$\therefore \frac{df}{dp} = \frac{9}{(0.1+p)^2} > 0 \quad \text{for all } p \in [0,1].$$

So $f(p)$ is an increasing function of p for $p \in [0,1]$.
Thus to maximise long-term earnings, we should use the value of p that maximises $f(p)$, which is $p = 1$.



2a) $T = \begin{cases} 3+X & \text{w.p. } 0.8 \\ 3+X+Y & \text{w.p. } 0.2 \end{cases}$ where $X \sim \text{Poisson}(1)$, $Y \sim \text{Poisson}(2)$, X, Y indept.

So $\mathbb{E}T = 0.8(3+\mathbb{E}X) + 0.2(3+\mathbb{E}X+\mathbb{E}Y)$ Law of Total Expectation

$= 0.8(3+1) + 0.2(3+1+2)$ $X \sim Po(1)$, $Y \sim Po(2)$

$\therefore$ $\underline{\mathbb{E}T = 4.4}$ as stated.

b) $G_{X+Y}(s) = \mathbb{E}(s^{X+Y})$

$= \mathbb{E}(s^X s^Y)$

$= \mathbb{E}(s^X)\,\mathbb{E}(s^Y)$ X, Y independent

$= e^{(s-1)}\, e^{2(s-1)}$ given in question

$\therefore$ $\underline{G_{X+Y}(s) = e^{3(s-1)}}$ Valid for $s \in \mathbb{R}$.

This is the PGF of the Poisson (3) distribution,

so $\underline{X+Y \sim \text{Poisson}(3)}$.

c) $\mathbb{P}(T \geq 5 \mid D=0) = \mathbb{P}(3+X \geq 5)$ because $T \sim 3+X$ if $D=0$

$= \mathbb{P}(X \geq 2)$

$= 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1)$

$= 1 - \frac{1^0}{0!}e^{-1} - \frac{1^1}{1!}e^{-1}$ $(X \sim \text{Poisson}(1))$

$= 1 - 2e^{-1}$

$= \underline{\underline{0.264}}$

2c cont.) Likewise,

$\mathbb{P}(T \geq 5 \mid D=1) = \mathbb{P}(3+X+Y \geq 5)$ because $T \sim 3+X+Y$ if $D=1$

$= \mathbb{P}(X+Y \geq 2)$ where $X+Y \sim \text{Poisson}(3)$

$= 1 - \frac{3^0}{0!}e^{-3} - \frac{3^1}{1!}e^{-3}$

$= 1 - 4e^{-3}$

$= \underline{\underline{0.801}}$

2d) $\mathbb{P}(D=1 \mid T \geq 5) = \dfrac{\mathbb{P}(T \geq 5 \mid D=1)\,\mathbb{P}(D=1)}{\mathbb{P}(T \geq 5 \mid D=1)\,\mathbb{P}(D=1) + \mathbb{P}(T \geq 5 \mid D=0)\,\mathbb{P}(D=0)}$ (Bayes Thm)

$= \dfrac{0.801 * 0.2}{0.801 * 0.2 + 0.264 * 0.8}$ by (c)

$= \underline{\underline{0.431}}$

2e) We need $Var(T) = \mathbb{E}_D\{Var(T \mid D)\} + Var_D\{\mathbb{E}(T \mid D)\}$.

Now $T = \begin{cases} 3+X+Y & \text{if } D=1 \text{ (prob } 0.2) \\ 3+X & \text{if } D=0 \text{ (prob } 0.8) \end{cases}$

So $\mathbb{E}(T \mid D) = \begin{cases} 6 & \text{w.p. } 0.2 \\ 4 & \text{w.p. } 0.8 \end{cases}$ and $Var(T \mid D) = \begin{cases} 3 & \text{w.p. } 0.2 \\ 1 & \text{w.p. } 0.8 \end{cases}$

(using indep of X, Y).

So $Var(T) = \mathbb{E}\{Var(T \mid D)\} + Var\{\mathbb{E}(T \mid D)\}$

$= \{3 * 0.2 + 1 * 0.8\} + \{6^2 * 0.2 + 4^2 * 0.8 - [6*0.2 + 4*0.8]^2\}$

$= 1.4 + 20 - (4.4)^2$

$\therefore$ $\underline{\underline{Var(T) = 2.04}}$

3a) $U = \begin{cases} 1 & \text{w.p. } 3/4 \\ 1 + U' + U'' & \text{w.p. } 1/4, \text{ where } U' \sim U'' \sim U,\ U', U'' \text{ independent.} \end{cases}$

So $H_u(s) = \mathbb{E}(s^U)$

$= \frac{3}{4}s + \frac{1}{4}s\,\mathbb{E}(s^{U'})\,\mathbb{E}(s^{U''})$ because U', U'' indep't

$\therefore\ H_u(s) = \frac{3}{4}s + \frac{1}{4}s\left[H_u(s)\right]^2$ because $U' \sim U'' \sim U$

$\Rightarrow\ \underline{s H_u(s)^2 - 4 H_u(s) + 3s = 0}$

Solving $\Rightarrow\ H_u(s) = \dfrac{4 \pm \sqrt{16 - 4 * 3s^2}}{2s}$

$\therefore\ H_u(s) = \dfrac{4 \pm \sqrt{16 - 12s^2}}{2s}$ as stated.

b) $\lim_{s\to 0} H_u(s) = \mathbb{P}(U = 0) = 0$ because it takes ≥ 1 step to reach state 1 from state 0.

Under the (+) root, $\lim_{s \to 0} \dfrac{4 + \sqrt{16 - 12s^2}}{2s} \neq 0$ (undefined).

So the (+) root can not be correct.

c) U is defective $\Leftrightarrow H_u(1) < 1$.

Now $\underline{H_u(1) = \dfrac{4 - \sqrt{16 - 12}}{2} = 1.}$

So $\underline{U \text{ is not defective.}}$

Thus $\underline{\mathbb{P}(\text{never reach } 1 \mid \text{start at } 0) = \mathbb{P}(U = \infty) = 0.}$

3d) $\mathbb{P}(\text{never reach state } -1 \mid \text{start at } 0) = \mathbb{P}(V = \infty)$

$= 1 - H_v(1)$

$= 1 - \left\{ \dfrac{4 - \sqrt{16 - 12}}{6} \right\}$

$= \underline{\dfrac{2}{3}}$

e) $T = \begin{cases} 1 + V & \text{w.p. } 3/4 \\ 1 + U & \text{w.p. } 1/4 \end{cases}$

So $G(s) = \mathbb{E}(s^T) = \frac{3}{4}s H_v(s) + \frac{1}{4}s H_u(s)$

$\Rightarrow\ G(1) = \frac{3}{4}H_v(1) + \frac{1}{4}H_u(1)$

$= \frac{3}{4} * \frac{1}{3} + \frac{1}{4} * 1$ from (c) & (d)

$\therefore\ \underline{G(1) = \frac{1}{2}}$

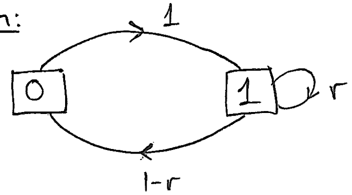
So $\underline{\mathbb{P}(\text{never return to } 0 \mid \text{start at } 0) = 1 - G(1) = \frac{1}{2}}$.

f) <u>Acceptable answer:</u> None of the calculations in (c), (d), or (e) are valid now, because they all used the translation-invariance of the process which is now violated by the boundary.

<u>Alternative acceptable answer:</u> Although we used translation-invariance to derive $H_u(s)$, the boundary does not affect the probability in (c): it is still true that the process is bound to reach state 1, starting from state 0.

However, the boundary does affect the probabilities in (d) and therefore (e), which are now invalid. Starting at 0, the new process can get stuck at 100 before it returns to 0 or reaches -1.

4a) Diagram:



Matrix:

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1-r & r \end{array}$$

Equilibrium equations:

Require $\underset{\sim}{\pi}$ s.t. $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$ and $\pi_0 + \pi_1 = 1$:

$$\Rightarrow \quad \pi_1 (1-r) = \pi_0 \quad (1)$$
$$\pi_0 + \pi_1 = 1 \quad (2)$$

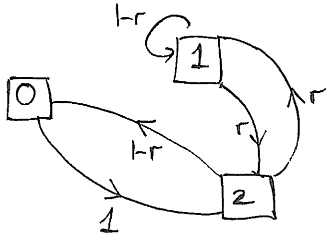
(2) in (1) $\Rightarrow \quad \pi_1(1-r) = 1 - \pi_1 \quad \Rightarrow \quad \underline{\pi_1 = \frac{1}{2-r}}$

so also $\underline{\pi_0 = 1 - \pi_1 = \frac{1-r}{2-r}}$

$\therefore$ Equilibrium distn is $\underset{\sim}{\pi}^T = (\pi_0, \pi_1) = \left(\frac{1-r}{2-r}, \frac{1}{2-r}\right)$

b) When $N=2$: states are 0, 1, 2.

Diagram



$$\text{so } P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 1-r & r \\ 2 & 1-r & r & 0 \end{array}$$

Given $\underset{\sim}{\pi}^T = \frac{1}{3-r}(1-r, 1, 1)$: note $\pi_0 + \pi_1 + \pi_2 = 1$ as needed.

Check $\underset{\sim}{\pi}^T P = \underset{\sim}{\pi}^T$:

$$\text{LHS} = \frac{1}{3-r}(1-r, 1, 1)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1-r & r \\ 1-r & r & 0 \end{pmatrix} = \frac{1}{3-r}(1-r, 1, 1) = \text{RHS}.$$

So $\underset{\sim}{\pi}^T$ as given is an equilibrium distn for this chain.

4c) If there are X_t umbrellas available at time t, then there are $N - X_t$ in the other location at time t.

If $X_t = 0$: no choice to carry umbrella, so $X_{t+1} = N$ w.p. 1.

If $X_t > 0$: no rain at $t \Rightarrow$ all umbrellas stay still $\Rightarrow X_{t+1} = N - X_t$

: rain at $t \Rightarrow$ one umbrella carried to other place :

$\Rightarrow X_{t+1} = N - X_t + 1$

$$\text{So} \quad X_{t+1} \mid X_t = \begin{cases} N & \text{w.p. } 1 \text{ if } X_t = 0 \\ N - X_t & \text{w.p. } 1-r \text{ if } X_t > 0 \\ N - X_t + 1 & \text{w.p. } r \text{ if } X_t > 0 \end{cases}$$

$$\Rightarrow \mathbb{E}(s^{X_{t+1}} \mid X_t) = \begin{cases} s^N & \text{if } X_t = 0 \\ (1-r)s^{N-X_t} + r s^{N-X_t+1} & \text{if } X_t > 0 \end{cases} \quad (\alpha)$$

Now suppose $X_t \sim \underset{\sim}{\pi}_*$.

$$\text{Then } \underbrace{\mathbb{E}(s^{X_t}) = \sum_{x=0}^{N} \pi_x s^x = \frac{1}{N+1-r}\left\{(1-r)s^0 + \sum_{x=1}^{N} s^x\right\}}_{\text{by definitions.}} \quad (*)$$

We want to show $X_{t+1} \sim X_t$: do this by showing $\mathbb{E}(s^{X_{t+1}}) = \mathbb{E}(s^{X_t})$

LHS $= \mathbb{E}(s^{X_{t+1}})$

$= \mathbb{E}_{X_t}\{\mathbb{E}(s^{X_{t+1}} \mid X_t)\}$ Law of total expectation

$= \mathbb{P}(X_t = 0)\,\mathbb{E}(s^{X_{t+1}} \mid X_t = 0) + \sum_{x=1}^{N} \mathbb{P}(X_t = x)\,\mathbb{E}(s^{X_{t+1}} \mid X_t = x)$

$= \pi_0 s^N + \sum_{x=1}^{N} \pi_x \left[(1-r)s^{N-x} + r s^{N-x+1}\right]$ using $X_t \sim \underset{\sim}{\pi}_*$ and eqn (α)

$= \frac{(1-r)s^N}{N+1-r} + \frac{1}{N+1-r}\left[(1-r)\{s^{N-1} + \dots + s^0\} + r\{s^N + \dots + s^1\}\right]$

4c cont.)

$$\Rightarrow \mathbb{E}(s^{X_{t+1}}) = \frac{1}{N+1-r}\left\{ (1-r)s^0 + (1-r+r)(s^N + \dots + s^1) \right\}$$

$$= \frac{1}{N+1-r}\left\{ (1-r)s^0 + \sum_{x=1}^{N} s^x \right\}$$

$$= \mathbb{E}(s^{X_t}) = \text{RHS} \quad \text{as specified in eqn } (*).$$

So if $X_t \sim \underline{\pi}_*$ we have shown $\mathbb{E}(s^{X_{t+1}}) = \mathbb{E}(s^{X_t})$,

so $X_{t+1} \sim \underline{\pi}_*$ also.

Because $X_{t+1} \sim X_t$ if $X_t \sim \underline{\pi}_*$, so $\underline{\pi}_*$ is the equilibrium dist.

The chain converges to equilibrium because it is irreducible (all #'s umbrellas are eventually accessible) and aperiodic (given).

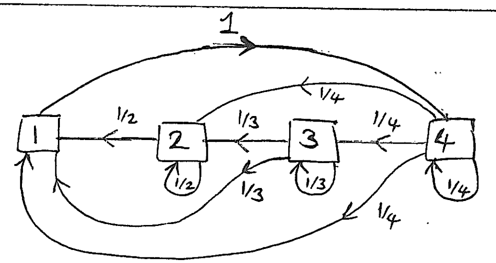
So $\mathbb{P}(\text{gets wet}) = \mathbb{P}(\text{rains and no umbrella})$

$= r\pi_0$ in the long term, by convergence

$$\mathbb{P}(\text{gets wet}) = \frac{r(1-r)}{N+1-r}.$$

5a)

Diagram



b) Yes, the chain does converge to equilibrium:
- it is irreducible, and the presence of loops implies aperiodic;
- finite state space implies an equilibrium distn exists.

5c) Using FSA, $m_{x+1} = 1 + \frac{1}{x+1}(m_1 + m_2 + \dots + m_{x+1})$ $(*)$. Valid for $x+1 = 2, \dots, N$

$$\Rightarrow \frac{x}{x+1} m_{x+1} = 1 + \frac{1}{x+1}(m_1 + \dots + m_x)$$

So $$m_{x+1} = \frac{x+1}{x} + \frac{1}{x}(m_1 + \dots + m_x) \quad (1)$$

But by the same argument,

$$m_x = 1 + \frac{1}{x}(m_1 + \dots + m_x) \quad \text{as in } (*)$$

Valid for $x = 2, \dots, N$

Substituting this in (1)

$$\Rightarrow m_{x+1} = \frac{x+1}{x} + (m_x - 1)$$

$$\therefore m_{x+1} = \frac{1}{x} + m_x \quad \text{as stated. Valid for } x = 2, \dots, N$$

Finally, $m_2 = 1 + \frac{1}{2}m_2 + \frac{1}{2}m_1$ using FSA directly

$\Rightarrow \frac{1}{2}m_2 = 1 + 0$ using $m_1 = 0$

$\therefore m_2 = 2.$

5d) Because the chain converges to equilibrium (part (b)), we seek π_1. By Theorem, we know $\pi_1 = \frac{1}{R_{11}}$ so we need R_{11}.

$R_{11} = \mathbb{E}(\text{time to return to state 1} \mid \text{start in state 1})$

$= 1 + m_N$ because from state 1 we take 1 step to state N (w.p. 1) then must return $N \to 1$, which is an average of m_N steps by defn.

$= 1 + \frac{1}{N-1} + m_{N-1}$ by (c)

$= 1 + \frac{1}{N-1} + \frac{1}{N-2} + m_{N-2}$ ... (c) again

$\vdots$

$= 1 + \frac{1}{N-1} + \frac{1}{N-2} + \dots + \frac{1}{2} + m_2$

$\therefore R_{11} = 2 + \sum_{r=1}^{N-1} \frac{1}{r}$ because $m_2 = 2$ from (c).

So the answer required is $\pi_1 = R_{11}^{-1} = \left\{ 2 + \sum_{r=1}^{N-1} \frac{1}{r} \right\}^{-1}$ as stated.

6a) $G(s) = \mathbb{E}(s^Y) = \sum_{y=0}^{\infty} s^y \,\mathbb{P}(Y=y)$

$$= \sum_{y=0}^{\infty} s^y \left(\tfrac{1}{2}\right)^{y+1}$$

$$= \tfrac{1}{2} \cdot \sum_{y=0}^{\infty} \left(\tfrac{s}{2}\right)^y$$

$$= \tfrac{1}{2} \cdot \frac{1}{1 - s/2} \quad \text{for} \quad \left|\tfrac{s}{2}\right| < 1$$

$$\therefore\ G(s) = \frac{1}{2-s} \quad \text{for} \quad |s| < 2, \quad \text{as stated.}$$

b) We need the smallest solution ≥ 0 to $G(s) = s$:

$$G(s) = \frac{1}{2-s} = s \Rightarrow 1 = 2s - s^2$$

$$\Rightarrow s^2 - 2s + 1 = 0$$

$$\Rightarrow (s-1)^2 = 0$$

$$\Rightarrow s = 1 \quad \text{only root.}$$

So $\gamma = \mathbb{P}(\text{ultimate extinction}) = 1$ (definite).

c) $G_{n+1}(s) = G_n\left(G(s)\right)$.

d) RTP $G_n(s) = \dfrac{n - (n-1)s}{(n+1) - ns}$ for $n = 1, 2, 3, \ldots$ ⊛

Base case: $n = 1$.

LHS ⊛ $= G_1(s) = G(s) = \dfrac{1}{2-s}$ by (a).

RHS ⊛ $= \dfrac{1 - 0s}{2 - 1s} = \dfrac{1}{2-s} =$ RHS ⊛.

So ⊛ is proved for case $n = 1$.

6d cont.) General case:

Suppose ⊛ is true for $n = x$, so we can assume

$$G_x(s) = \frac{x - (x-1)s}{x+1 - xs} \qquad ⓐ$$

RTP ⊛ is true for $n = x+1$, i.e.

$$\text{RTP} \quad G_{x+1}(s) = \frac{x+1 - xs}{x+2 - (x+1)s} \qquad (**)$$

LHS $(**)$ $= G_{x+1}(s)$

$= G_x\left(G(s)\right)$ by (c)

$= \dfrac{x - (x-1)G(s)}{x+1 - x\,G(s)}$ by allowed ⓐ

$= \dfrac{x - \frac{x-1}{2-s}}{x+1 - \frac{x}{2-s}}$ using $G(s) = \dfrac{1}{2-s}$ by part (a)

$= \dfrac{2x - sx - x + 1}{2x + 2 - sx - s - x}$ multiplying through by 2-s

$= \dfrac{x+1 - xs}{x+2 - (x+1)s}$

$=$ RHS $(**)$.

So ⊛ is proved for $n = 1$, and if true for $n = x$ it is proved for $n = x+1$. So ⊛ is proved for all $n = 1, 2, 3, \ldots$ □

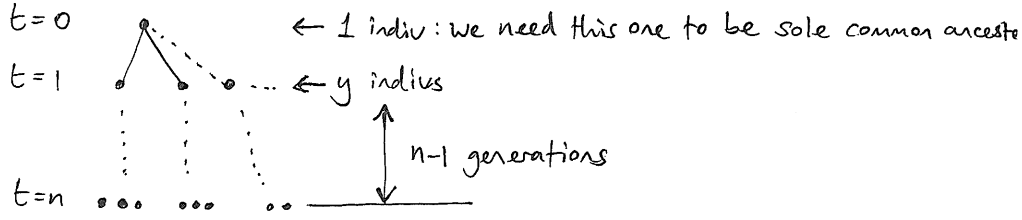
e i) $\gamma_n = \mathbb{P}(Z_n = 0) = G_n(0) = \dfrac{n}{n+1}$

ii) $e_n = \mathbb{P}(Z_n > 0) = 1 - \mathbb{P}(Z_n = 0) = 1 - \gamma_n = \dfrac{1}{n+1}$.

6f) Partition over family size Y in generation 1:

$$\mathbb{P}(Z_n > 0 \text{ and } T_n = 0) = \sum_{y=0}^{\infty} \mathbb{P}(Z_n > 0 \cap T_n = 0 \cap Y = y). \quad (*)$$

Consider the conditions needed for $T_n = 0$ and $Z_n > 0$:



The only way the MRCA can be at time $t=0$ is if it has at least 2 offspring at time 1 who both have surviving descendents at time n.

→ If no indivs at time 1 have surviving descendents at time n, then $Z_n = 0$ (not allowed);

→ If exactly one indiv at time 1 has surviving descendents at time n, then this indiv at $t=1$ would be a MORE recent common ancestor than the indiv at $t=0$ (not allowed).

Define r.v. X to be the number of indivs at time $t=1$ who have surviving descendents at time $t=n$ ($n-1$ generations later).

Then $X \mid Y \sim \text{Binomial}(Y, e_{n-1})$. Y trials (indivs @ $t=1$) each with prob $= e_{n-1}$ of having surviving descendents.

We need
$$\mathbb{P}(X \geq 2 \mid Y=y) = 1 - \mathbb{P}(X=0 \mid Y=y) - \mathbb{P}(X=1 \mid Y=y)$$
$$= 1 - (1-e_{n-1})^y - y(1-e_{n-1})^{y-1} e_{n-1}$$
$$= \underline{1 - \gamma_{n-1}^y - y\gamma_{n-1}^{y-1} e_{n-1}}. \quad (**)$$

6f cont.) Reverting to $(*)$:

$$\mathbb{P}(Z_n > 0 \cap T_n = 0) = \sum_{y=0}^{\infty} \mathbb{P}(Z_n > 0 \cap T_n = 0 \mid Y=y)\, \mathbb{P}(Y=y)$$
$$= \sum_{y=0}^{\infty} \left(\tfrac{1}{2}\right)^{y+1} \mathbb{P}(X \geq 2 \mid Y=y)$$
$$= \sum_{y=2}^{\infty} \left(\tfrac{1}{2}\right)^{y+1} \left\{1 - \gamma_{n-1}^y - y\gamma_{n-1}^{y-1} e_{n-1}\right\} \text{ by } (**)$$

note: summing from $y=2$ now because $\mathbb{P}(X\geq 2 \mid Y=y) = 0$ for $y=0$ or $y=1$

$$\therefore \underline{\underline{\mathbb{P}(Z_n > 0 \cap T_n = 0) = \sum_{y=2}^{\infty} \left(\tfrac{1}{2}\right)^{y+1} \left\{1 - \left(\tfrac{n-1}{n}\right)^y - y\left(\tfrac{n-1}{n}\right)^{y-1}\left(\tfrac{1}{n}\right)\right\}}} \quad (✡)$$

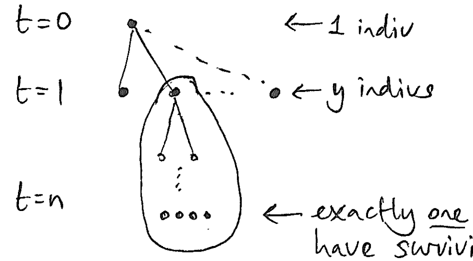
as stated, using γ_{n-1} and e_{n-1} in (e).

So
$$\alpha_{n,0} = \mathbb{P}(T_n = 0 \mid Z_n > 0)$$
$$= \frac{\mathbb{P}(T_n = 0 \cap Z_n > 0)}{\mathbb{P}(Z_n > 0)}$$
$$= \frac{1}{e_n} \mathbb{P}(T_n = 0 \cap Z_n > 0) \quad \text{because } \mathbb{P}(Z_n > 0) = e_n \text{ by defn.}$$
$$\underline{\underline{\alpha_{n,0} = (n+1)\, \mathbb{P}(T_n = 0 \text{ and } Z_n > 0)}} \quad \text{with } (✡) \text{ to complete the expression.}$$

Finally, consider $\alpha_{n,1} = \mathbb{P}(T_n = 1 \mid Z_n > 0)$.

i.e.
$$\alpha_{n,1} = \frac{1}{e_n} \mathbb{P}(T_n = 1 \cap Z_n > 0)$$
$$\underline{\alpha_{n,1} = \frac{1}{e_n} \sum_{y=0}^{\infty} \mathbb{P}(T_n = 1 \cap Z_n > 0 \mid Y=y)\, \mathbb{P}(Y=y).}$$

6f cont.) For $T_n = 1$ we need:



← exactly one of the y indivs at time $t=1$ must have surviving descendents, AND this indiv at $t=1$ must be the MRCA over the $n-1$ generation of its own tree.

So

$$\alpha_{n,1} = \frac{1}{\ell_n} \sum_{y=0}^{\infty} \left(\frac{1}{2}\right)^{y+1} \mathbb{P}(X=1 \mid Y=y)\, \mathbb{P}\left(\begin{array}{l}\text{single } (n-1)\text{-generation line has} \\ \text{MRCA in gen } 0, \text{ given it survives}\end{array}\right)$$

$$= \frac{1}{\ell_n} \sum_{y=1}^{\infty} \left(\frac{1}{2}\right)^{y+1} y\, \ell_{n-1}\, \gamma_{n-1}^{y-1}\, \alpha_{n-1,0}$$

summing $y=1$ now because $\mathbb{P}(X=1 \mid Y=0) = 0$.

$$\therefore \quad \alpha_{n,1} = (n+1) \sum_{y=1}^{\infty} \left(\frac{1}{2}\right)^{y+1} y \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^{y-1} \alpha_{n-1,0}\,.$$